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FORCED VIBRATIONS OF SYSTEMS WITH NONLINEAR RESTORING FORCE*

BY

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1. Introduction. The purpose of our lecture is, briefly stated, the discussion of periodic solutions of the differential equation

$$\ddot{x} + f(x) = F \cos \omega t, \tag{1}$$

in which $f(x)$ is in general a nonlinear function of x. For $f(x) = g/l \sin x$ we have, for example, the case of the simple pendulum with a periodic external force. Such a differential equation may occur in electrical problems when iron is contained in the magnetic circuit.[1] Any system with one degree of freedom[2] and consisting of a mass and an elastic restoring force will lead to a differential equation of the type (1) if the displacement x is not kept very small.

The problems to be discussed here differ essentially from those of self-excited oscillations where *negative damping* is involved, as is the case for the group of problems first treated by van der Pol. Neither the nonlinear character of the restoring force as such nor linearity or nonlinearity in possible friction forces is responsible for the differences. In the case of self-excited oscillations results are altered but little if a nonlinear restoring force is assumed, while the results to be discussed here would not be altered in general character if nonlinear *positive damping* were assumed. In other words, it appears that even a slight negative damping will dominate the entire phenomena.

Our discussion will yield, for the most part, results which are already well known. Much of the material can be found in the standard textbooks, e.g., in the books of Timoshenko [28] and den Hartog [5] cf. also [11, 20]. How-

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[1] For this subject see [32] and the list of references given there.

[2] Some cases of free oscillations with two degrees of freedom have been treated [4] and [12] (3. Mitt.). For experimental results in a case with infinitely many degrees of freedom, see [22].

ever, the problems are of sufficient intrinsic importance that they merit thorough consideration and discussion.

We are interested, to begin with, in obtaining insight into the physical phenomena connected with the case under discussion. It turns out that it is highly useful to begin by obtaining information of a *qualitative* character. Once this has been done, it is not very difficult to see how one should proceed to obtain accurate quantitative information. The validity of the methods to be discussed here could be justified rigorously in most cases, but we do not wish to take up these matters here. We prefer to stress methods which lead to the essential qualitative information in as simple a manner as possible. What might be found original in this lecture consists largely in the exploitation of this point of view.

2. Nonlinear restoring forces. The differential equation (1) has been treated for a considerable number of different restoring forces $f(x)$, cf. [5, 6, 7, 16, 24, 25, 28]. Some of these cases are indicated in fig. 1:

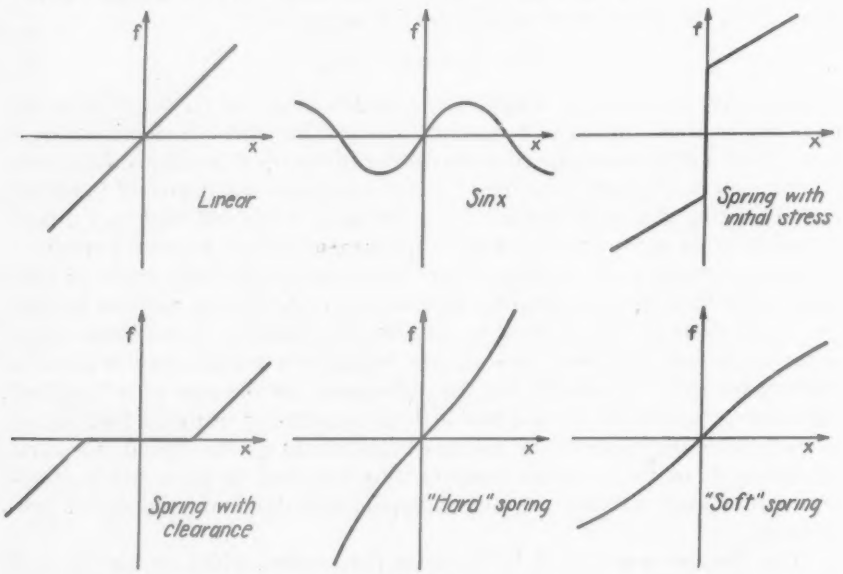

FIG. 1

The forces indicated in fig. 1 are *symmetrical*, i.e. $f(-x) = -f(x)$. This need not always be the case, of course; but we shall assume it in what follows. The last two examples, for the *hard* and *soft* springs, indicate the essential distinction between different types of springs, i.e., those for which the *stiffness* $f'(x)$ is an increasing or a decreasing function of x. The first case is typified by a mass attached to a stretched string, the second by the pendulum.

Since methods and also qualitative results appear not to depend greatly upon the special form of $f(x)$, we shall choose for this function always the expression

$$f(x) \equiv \alpha x + \beta x^3, \qquad \alpha > 0.$$

In any case there is no great loss in generality involved for moderate values of x, since our expression might be regarded as the linear and cubic terms in the power series for $f(x)$. Note also that $\beta > 0$ characterizes a hard spring (stretched string), while $\beta < 0$ characterizes a soft spring (pendulum).

The differential equation (first treated by Duffing [8]) which we want to consider in the following is, thus,

$$\ddot{x} + (\alpha x + \beta x^3) = F \cos \omega t. \tag{2}$$

3. Periodic solution. We seek periodic solutions of (2) with frequency ω. (The term frequency will be used here in place of the more correct term circular frequency, since no confusion is likely to result). That solutions of (2) other than periodic ones exist is certainly true (even unbounded ones exist for $\beta < 0$). However, the literature of the subject is almost entirely devoted to the periodic solutions, which are the interesting ones in the engineering applications. Apparently the experimenters always find periodic motions, at least after some transient motions have died out. Viscous damping, which is always present in any actual case, seems to act in such a way that the motions in a wide variety of cases tend, as $t \to \infty$, to periodic ones. It would be of considerable interest to prove that the solutions of (2) with a damping term added (and under appropriate conditions) are of this character.

We turn, then, to the problem of finding periodic solutions of (2). No explicit solutions of this differential equation are known and we are forced to turn to approximate methods. Perhaps the simplest of these is the iteration method. Let us write (2) in the form

$$\ddot{x} = -(\alpha x + \beta x^3) + F \cos \omega t \tag{3}$$

and insert

$$x_0 = A \cos \omega t \tag{4}$$

as a first approximation in the right-hand side. This means, in effect, that we assume β to be small so that a motion not greatly different from a simple harmonic motion can be expected. In fact, we assume in all of our methods but one that β is small. Upon using the identity

$$\cos^3 \omega t \equiv \tfrac{3}{4} \cos \omega t + \tfrac{1}{4} \cos 3\omega t \tag{5}$$

we find

$$\ddot{x}_1 = -(\alpha A + \tfrac{3}{4}\beta A^3 - F) \cos \omega t - \tfrac{1}{4}\beta A^3 \cos 3\omega t \tag{6}$$

as equation for the next approximation x_1. Integrating this one obtains

$$x_1 = \frac{1}{\omega^2}(\alpha A + \tfrac{3}{4}\beta A^3 - F)\cos \omega t + \frac{1}{36}\frac{\beta A^3}{\omega^2}\cos 3\omega t, \tag{7}$$

the integration constants being taken as zero to insure that x_1 and the next following approximation x_2 be periodic.

So far this is quite straightforward. What to do from now on is not so clear. One might seek to continue the process by inserting x_1 in the right side of (3) in order to find an x_2, etc., which would be a rather natural procedure. The significance of this procedure can be illustrated by the following figure which indicates the well known *response* curve for the linear forced vibration, i.e., a curve showing the amplitude $|A|$ of the steady forced vibration as a function of the frequency ω, the force amplitude F being a parameter (fig. 2):

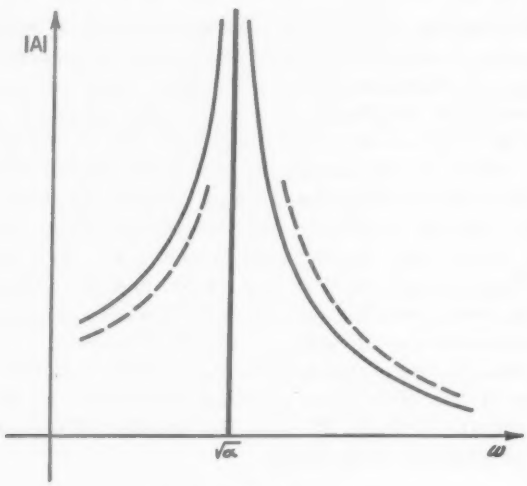

FIG. 2

The procedure just outlined would mean that ω is held fixed while $|A|$ is left open, for $\beta \neq 0$. This would yield certain curve (shown dotted) in the neighborhood of the linear response curve. It is clear that there would be difficulties near $\omega = \sqrt{\alpha}$. But worst of all, the really essential features of the response curves for $\beta \neq 0$ would not be obtained at all.

The method used by Duffing [8], who seems to have been the first to obtain the significant results for the differential equation (2) was as follows: the coefficient A_1 of $\cos \omega t$ in (7) is taken equal to A in (4), on the ground that A_1 should differ but little from A if (4) is truly a reasonable first approximation. Also, Duffing argues, such a procedure would furnish the exact result in the linear case ($\beta = 0$) and might hence be expected to yield good results

for β small, which we assume here anyway so that convergence can be expected. This reasoning of Duffing leads thus to what proves to be the decisive relation (for $A \neq 0$)

$$\omega^2 = \alpha + \tfrac{3}{4}\beta A^2 - \frac{F}{A} \tag{8}$$

between the *amplitude* A and frequency ω of the periodic solution. The relation (8) has been written purposely so that ω^2 is given as a function of A: as we shall see it is decisive to consider A as a prescribed quantity in terms of which ω is to be determined.

Before discussing the significance and interpretation of (8), let us first obtain the same relation by a procedure which is perhaps a more systematic one. Equation (3) is modified by adding a term $\omega^2 x$ to both sides:

$$\ddot{x} + \omega^2 x = -\left[(\alpha - \omega^2)x + \beta x^3\right] + F \cos \omega t. \tag{9}$$

As a first approximation x_0 to a periodic solution we begin with the solution of (9) for $\alpha = \omega^2$, $\beta = 0$, $F = 0$, i.e., with a free undamped linear oscillation. This leads at once to

$$x_0 = A \cos \omega t, \tag{10}$$

with A arbitrary. Upon insertion in the right-hand side of (9) and use of (5) once more we find

$$\ddot{x}_1 + \omega^2 x_1 = \left\{(\omega^2 - \alpha)A - \tfrac{3}{4}\beta A^3 + F\right\} \cos \omega t - \tfrac{1}{4}\beta A^3 \cos 3\omega t. \tag{11}$$

We require always that the solution be periodic; hence it is necessary that the coefficient P_1 of $\cos \omega t$ in the right hand side of (11) should vanish in order to avoid the resonance case and hence the occurrence of nonperiodic terms in the solution of (11). The vanishing of this coefficient yields, obviously, the same relation as (8), which is to be regarded as an equation to determine ω after A has been prescribed. Once this relation has been satisfied, the solution of (11) will be

$$x_1 = A_1 \cos \omega t + B_1 \sin \omega t + \frac{\beta A^3}{32\omega^2} \cos 3\omega t, \tag{12}$$

in which A_1 and B_1 are arbitrary. The method of fixing A_1 and B_1 is, however, now clear. We simply set $A_1 = A$ and $B_1 = 0$, the value of ω being left open. If we stop with the degree of approximation implied in x_1 we should have, then, as approximate solution

$$x = A \cos \omega t + \frac{1}{32} \frac{\beta A^3}{\alpha + \tfrac{3}{4}\beta A^2 - \dfrac{F}{A}} \cos 3\omega t. \tag{13}$$

It is perhaps worth while to consider how the iterations should proceed: x_1 from (13) is inserted again in the right-hand side of (9). A term $P_2 \cos \omega t$ will again occur and we must require, as before, that $P_2 = 0$. The equation $P_2 = 0$ yields an improved relation between ω and A that takes the place of (8). After integration the coefficient of $\cos \omega t$ is again prescribed to have the value A while the coefficient of $\sin \omega t$ is taken as zero.

One could interpret this method as meaning that the first coefficients in the Fourier series for $x(t)$ are prescribed once and for all to be A and zero, while the frequency is left open. In what follows it is convenient to speak of the quantity A as the *amplitude* of the oscillation, though it should be referred to rather as the first Fourier coefficient.

Our procedures, in any case, require that A be fixed while ω, the frequency, is to be determined by relation (8). Why this at first sight seemingly unnatural procedure leads to the desired results can be best seen through a discussion of relation (8), which yields a set of curves, the *response* curves (in first approximation), with the force amplitude F as parameter. Usually the sign of A is not considered essential (it means a phase shift) so that only $|A|$ is plotted. The curves for $|A|$ against ω are then readily seen to be of the form shown in fig. 3, where curves for $\beta > 0$, $\beta = 0$ and $\beta < 0$ are given:

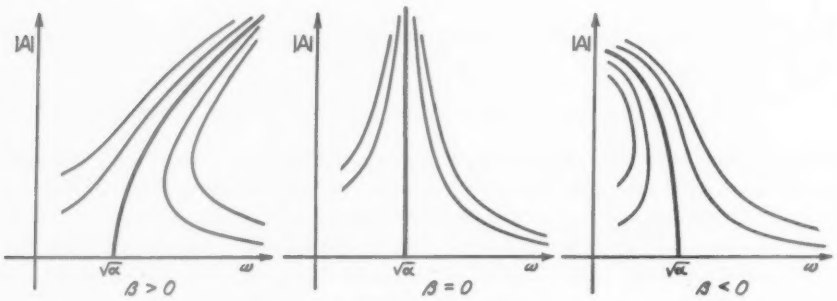

FIG. 3

The interpretation of the effect of the nonlinearity on the response curves is quite clear: The entire family of response curves (with F as parameter) for the linear forced oscillations is bent left or right depending upon whether the spring is a soft or a hard one. That these curves mirror the essential facts, at least for β and A not too large, cannot be doubted.

We can now see the reasons why A, instead of ω, should be prescribed. To begin with, we started our approximation with $\omega = \sqrt{\alpha}$, $\beta = 0$, $F = 0$, that is, at point 1 in fig. 4, which superimposes curves for $\beta = 0$ upon curves for $\beta > 0$. One sees that, for $\beta = 0$, A is completely arbitrary while ω is fixed and it is therefore necessary to prescribe A but leave ω open for $\beta \neq 0$.

The further steps in the iterations correspond to the passage from this state to one indicated by point 2 in fig. 4. One sees also that to hold ω fixed while varying A could yield at most one of the branches of the response curves.

This discussion indicates that the guiding principle to be followed independent of the particular scheme of approximation used is to observe what is arbitrary and what is fixed in the linear case, i.e., for $\beta=0$. For $\beta\neq 0$, any quantity that is arbitrary for $\beta=0$ should be prescribed, and vice versa.

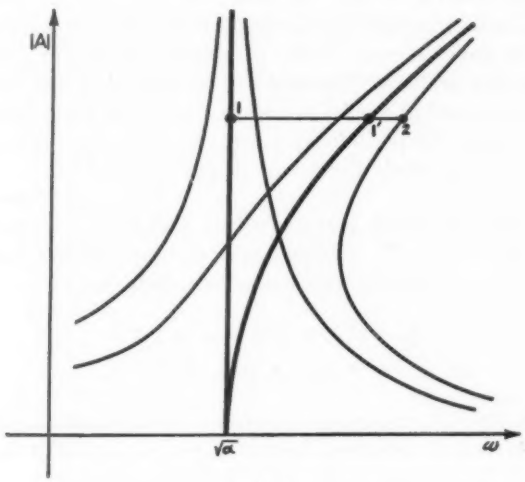

FIG. 4

4. Other methods. Before proceeding to draw the interesting physical conclusions from the response curves, we prefer to discuss some other methods of approximating the periodic solutions of (2).

One of the best known and most often used methods for nonlinear problems is the perturbation method, cf. [28], which consists for our problems in developing the solution $x(t)$ in powers of a parameter, say ϵ, with coefficients prescribed to be periodic functions of t:

$$x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots . \tag{14}$$

However, as we have seen, the frequency should not be prescribed but rather left open, and this requires that ω should also be developed with respect to ϵ:

$$\omega = \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots , \tag{15}$$

where the ω_i are constants. It is awkward to work with functions which have

a variable period, but one can avoid this by introducing a new variable $\theta = \omega t$, so that (2) becomes

$$\omega^2 x'' + (\alpha x + \beta x^3) = F \cos \theta, \tag{16}$$

the prime meaning differentiation with respect to θ. The functions x_i in (14) can now be prescribed to have the fixed period 2π. The perturbation series (14) and (15) are now to be inserted in (16), and the coefficients of the powers of ϵ in the resulting series are equated to zero. The result is a set of linear differential equations for the x_i. The parameter ϵ is to a certain degree arbitrary. A very reasonable choice for it would be to take $\epsilon = \beta$, so that (14) could be considered a development in the neighborhood of the solutions of the linearized vibration problem. The first few steps lead to essentially the same result as those obtained above by iteration. One finds, for example, that relation (8) results from the periodicity requirement. It should be said that the convergence of the perturbation series could be established with no great difficulty.

Since the solutions which interest us are periodic ones, they will possess Fourier series developments. This suggests writing the solutions $x(t)$ at the outset as such a series with undetermined coefficients:

$$\begin{aligned} x = A_0 &+ A_1 \cos \omega t + A_2 \cos 2\omega t + \cdots \\ &+ B_1 \sin \omega t + B_2 \sin 2\omega t + \cdots . \end{aligned} \tag{17}$$

Upon insertion into (2) one obtains an infinite set of nonlinear algebraic equations for the A_i and B_i which in their turn are to be solved by iterations or perturbations. Our previous experience indicates, however, that we should not prescribe ω but rather the first coefficients in (17) and this is decisive in fixing the correct set-up for iterations, say. Probably the Fourier series method is the simplest for actual computation once one has the clue to the correct procedure.

The approximation methods considered so far have all taken the solution for the linear free oscillation as a starting point. Although it is true that the significant qualitative features of the response curves can thus be obtained, it is clear that other procedures could be expected to yield more accurate results. Instead of beginning with the free linear oscillation, for example, one might begin with the free nonlinear oscillation, that is, at point 1′ rather than point 1 in fig. 4. For a given value of the force amplitude F it is quite obvious that the solution corresponding to point 1′ is much nearer to the desired solution (point 2) than is point 1. It is, of course, quite feasible to begin with a solution corresponding to point 1′, i.e., with a solution of $\ddot{x} + f(x) = 0$, since such a differential equation can be solved by explicit integration. With this solution as a basis one might then proceed by the perturbation method using F as small parameter, as for example has been done

by G. Hamel [9] and R. Iglisch [12, 13]. The method of variation of parameters has also been used, for example, by E. W. Brown [4], and A. N. Krylov and N. Bogoliùboff [18, 19] have used such methods in a considerable number of papers.

The method of M. Rauscher [24, 25] is an iteration procedure which also takes the solution corresponding to point $1'$ as its starting point. It is convenient here also to introduce the variable $\theta=\omega t$ in order to work with functions which have a fixed period (as in the perturbation method). With this variable our differential equation is

$$\omega^2 x'' + f(x) = F\cos\theta. \tag{18}$$

The idea of Rauscher is to take as first approximation the periodic solution $x_0(\theta)$ of

$$\omega_0^2 x'' + f(x) = 0 \tag{19}$$

which has a prescribed amplitude A (i.e., $(x_0)_{\max}=A$). The quantity ω_0 is to be determined in such a way that x_0 will have the period 2π. This is a problem that can be solved by explicit integration, which may be carried out graphically if necessary. The quantity ω_0 is, of course, taken as a first approximation to the unknown frequency ω. The essentially new idea of Rauscher consists then in replacing θ in the right hand side of (18) by the function $\theta_0(x)$ obtained by inverting the solution $x_0(\theta)$ of (19), over a half period in which x_0 is a monotone function of θ. This yields as equation for the next approximation:

$$\omega_1^2 x'' + f(x) - F\cos\left[\theta_0(x)\right] = 0, \tag{20}$$

which is an equation of the same type as (19): it differs only in that the restoring force is different. Its solution, for the same fixed A and same period 2π, yields a second approximation $x_1(\theta)$ and a new value ω_1 for ω, which will in general be different from ω_0. The general scheme of this iteration method thus consists in finding the solution $x_n(\theta)$ of period 2π of

$$\omega_n^2 x'' + f(x) - F\cos\left[\theta_{n-1}(x)\right] = 0 \tag{21}$$

for $(x_n)_{\max}=A$, where $\theta_{n-1}(x)$ has been obtained by inverting the solution of the equation for $(n-1)$. At the same time the nth approximation ω_n for ω is found. Actually, there is no need to invert the solution each time: The Rauscher method makes direct use of the fact that the solution of an equation of the type (19) is obtained most conveniently at the outset in the form $\theta=\theta(x)$. In the case of the pendulum, for example, the time is obtained as an elliptic integral which must be inverted to obtain the displacement as an elliptic function of the time.

The amount of numerical computation is not small in this method, but

the convergence is so rapid that it may well be the best method of obtaining accurate quantitative results. Rauscher finds that the second step usually furnishes a very close approximation. It might be objected that the method of Rauscher could not be used if viscous damping were present since the differential equation

$$x'' + cx' + f(x) = 0$$

cannot be integrated explicitly. However, the term cx' can be treated in much the same manner as the term $F \cos \theta$ once a phase shift has been introduced.

Finally, brief mention should be made of two other methods. The Ritz method has been used [8, 9, 27, 31]. It consists in minimizing the expression

$$\int_0^T \left\{ \dot{x}^2 - \alpha x^2 - \frac{\beta}{2} x^4 + 2xF \cos \omega t \right\} dt$$

with respect to periodic functions $x(t)$ of period $T = 2\pi/\omega$. The expression attains a minimum, however, only for soft springs ($\beta < 0$).

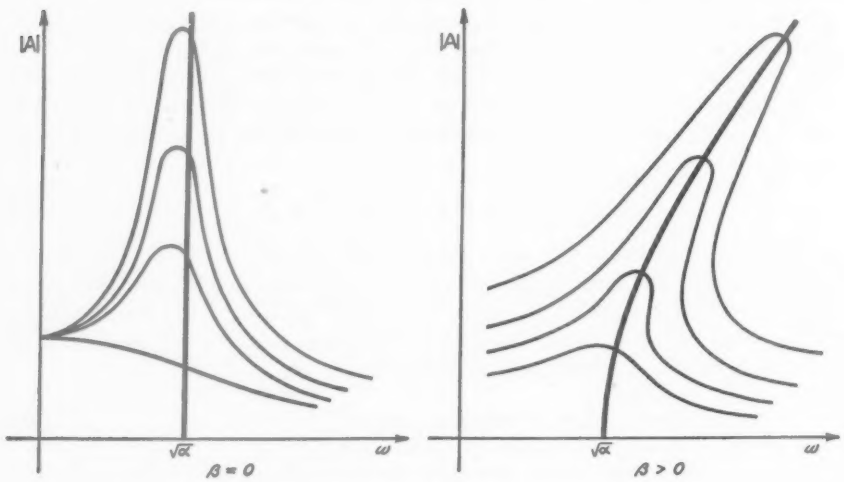

FIG. 5

In case one is interested in obtaining a solution satisfying given initial conditions (such a solution need not be periodic), the graphical method of E. Meissner [23] is to be recommended. The method is particularly well designed for just the type of differential equation considered here.

5. Jump phenomena. We turn now to the physical conclusions which can be drawn from the form of the response curves. Instead of discussing these

phenomena on the basis of the curves of fig. 3, we prefer to consider the analogous curves which result under the assumption of slight viscous damping in the system. By analogy with the response curves in the linear case with damping, one would expect the curves to be rounded off as indicated in fig. 5: This is a point to which we shall return a little later.

In the following fig. 6, we show a typical response curve (i.e., for a given value of the force amplitude F) for the case of a hard and a soft spring:

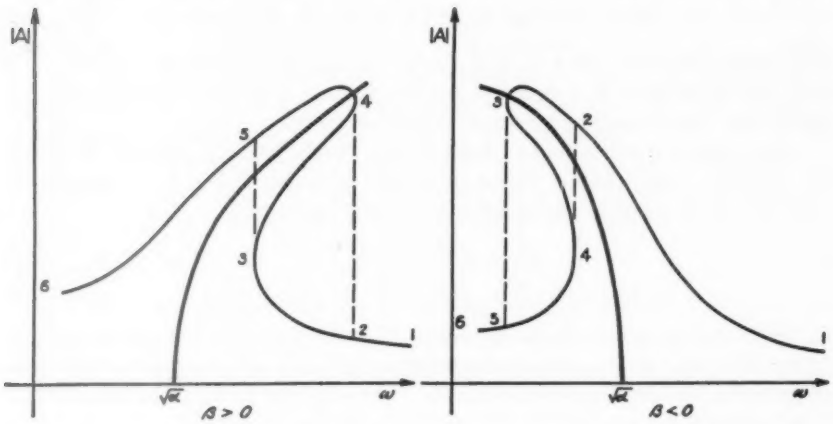

FIG. 6

Let us imagine an experiment performed in which the amplitude F of the external force is held constant, but its frequency ω is slowly varied. (If ω were held fixed and F slowly varied, the results would be of the same general character.) Consider first the case of a hard spring ($\beta>0$) and suppose that ω is rather large at the beginning of our experiment, i.e., we start at point 1 on the curve. As ω is decreased A slowly increases through point 2 until point 3 is reached. Since F is held constant, a further decrease in ω would require a jump from point 3 to point 5 with an accompanying increase in the amplitude A, after which A decreases with ω. Upon performing the experiment in the other direction, i.e., starting at point 6 and increasing ω, the amplitude follows the 6→5→4 portion of the curve, then jumps to point 2 and afterwards slowly decreases. The circumstances are quite similar with a soft spring, but the jumps in amplitude take place in the reverse direction. It would not have been necessary, we see now, to consider the influence of damping in order to conclude that a jump from point 3 to point 5 should take place (for $\beta>0$) on decreasing ω, but the jump from point 4 to point 2 on increasing ω would be inexplicable on the basis of the curves of fig. 3.

This curious behavior of nonlinear systems has often been observed by

experiment. Probably the first to discuss it experimentally was O. Martienssen [21]. Martienssen carried out his experiments with an electrical apparatus involving a condenser and an inductance with an iron core; he ascribed the results to the nonlinearity.

6. Effect of viscous damping. We wish to discuss briefly a method of treating the effect of viscous damping, i.e., the modifications which result upon adding a term $c\dot{x}$, $c>0$, to the left-hand side of (3):

$$\ddot{x} + c\dot{x} + (\alpha x + \beta x^3) = F \cos \omega t - G \sin \omega t. \tag{22}$$

The amplitude $H=\sqrt{F^2+G^2}$ of the external force is prescribed, but the ratio F/G is not prescribed in order to permit a phase difference between the applied force and the resulting periodic displacement.

The iteration method can be applied in much the same manner as above by beginning with $x_0=A \cos \omega t$ as a first approximation. In place of the relation (8), we now have the following two relations:

$$(\alpha - \omega^2)A + \tfrac{3}{4}\beta A^3 = F, \tag{23}$$

$$c\omega A = G. \tag{24}$$

Once can draw an interesting, though not unexpected, conclusion immediately. It is that no periodic motion except the state of rest can result if there is damping but no external force, i.e., if $F=G=0$ with $c\neq 0$. In fact, (24) is satisfied in this case only for $A=0$ which implies $x_0\equiv 0$.

Equations (23) and (24) are squared and added to obtain

$$[(\alpha - \omega^2)A + \tfrac{3}{4}\beta A^3]^2 + c^2\omega^2 A^2 = F^2 + G^2 = H^2, \tag{25}$$

which gives the relation between the amplitude A and frequency ω of the vibration which results from the force amplitude H. Equation (25) replaces (8) for this case; it of course reduces to (8) when $c=0$.

The discussion of the response curves furnished by (25) turns out to be quite easy and to yield the expected results. One finds that the curves have only one branch instead of two as in the undamped case, and that each curve shows a single maximum for A. The locus of the maxima (for different H) is a curve which runs close to the response curve for the free undamped oscillation ($H=c=0$), always remaining to the left of it. With no damping there is only one vertical tangent on each response curve. With damping a second vertical tangent appears; the new point of tangency lies near to the curve for free undamped oscillations. These results have already been indicated in figs. 5 and 6.

It is of some interest to note that the curves for values of H less than a certain limit value will have no vertical tangents, which means that for such amplitudes no jump phenomena would occur.

The effect of viscous damping on the response curves appears to have been

discussed first by E. V. Appleton [1]. Appleton uses the method of variation of parameters. He investigates also in detail the stability of the solutions. It is perhaps of interest to note that Appleton was led to the problem through observing the peculiar behavior of a certain galvanometer at the Cavendish laboratory at Cambridge. This behavior corresponds to what was called jump phenomena above.

7. Stability questions. We shall discuss briefly the stability question for the cases without damping. Probably the most difficult question in any consideration of dynamic stability is that of the definition of stability itself in a reasonable way. Into this question we do not wish to enter here. We take, simply, the following often used definition of stability: let $x(t)$ and $x(t)+\delta x(t)$ be two solutions of our differential equation. Consider the *variational equation* which results when $x+\delta x$ is inserted in the original differential equation and powers of δx above the first are neglected. If *all* solutions δx of this equation are bounded, then $x(t)$ is said to be stable, otherwise unstable.

In particular, the variational equation for Duffing's equation (2) is

$$\delta\ddot{x} + (\alpha + 3\beta x^2)\delta x = 0. \tag{26}$$

In the cases under discussion here $x(t)$ is a periodic function and thus (26) is a Hill's equation. The question is then whether for a given $x(t)$ all solutions of (26) are bounded or not. Or we might put the problem a little differently by referring to the A-ω-plane, the points of which characterize the periodic solutions $x(t)$. The problem thus is to divide this plane into regions which correspond to stable solutions and others which correspond to unstable ones. Questions of this kind have been discussed, cf. for example [1, 18, 29, 31], but the problem has apparently not been solved completely.

Some conclusions can, however, be reached without much difficulty. Consider, for example, fig. 7, which indicates the response curve for $F=0$ (i.e., for the free oscillation) together with the locus of vertical tangents on the response curves for $F>0$. (In connection with the latter curve see [12, 14, 15]). Our discussion of the jump phenomena would lead us to expect that the shaded area between these two curves corresponds to solutions which are unstable in some sense—quite possibly in the sense of the above definition. It can at least be shown that the two curves in question really do represent boundary curves between regions of stable and unstable solutions in the sense of the definition.

If one were to be contented with taking $A\cos\omega t$ as a sufficiently close approximation to $x(t)$, then equation (26) would be a Mathieu equation. For the special case of the Mathieu equation the problem of separating stable and unstable solutions has been completely solved. If one makes use of this theory it turns out that our two curves have contact of the second order at $\omega=\sqrt{\alpha}$, $|A|=0$ with curves delimiting a region of unstable solutions of the Mathieu equation, the unstable region being that within the cusp.

8. Subharmonic response. Up to now we have considered periodic solutions of (3) for which the frequency is always exactly the same as that of the external force $F \cos \omega t$. Permanent oscillations with a frequency $\frac{1}{2}, \frac{1}{3}, \cdots 1/n, \cdots$ of that of the applied force can, however, occur in nonlinear systems, in particular in our case of the Duffing equation. To this phenomenon the term subharmonic resonance is usually applied.[3]

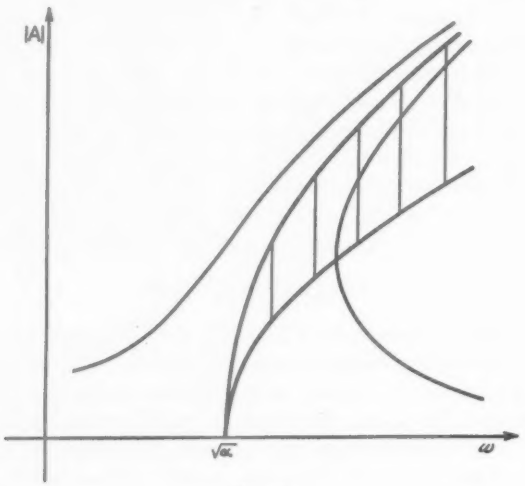

FIG. 7

The fact that subharmonic oscillations occur can hardly be denied since they have been so often observed. But it is not an entirely simple matter to give a plausible physical explanation for their occurrence. Let us recall the behavior of linear systems. If the frequency of the free oscillation of a linear system is ω/n (n an integer, say) then a periodic external force of frequency ω can excite the free oscillation in addition to the forced oscillation of frequency ω. But since some damping is always present the free oscillation is damped out so that the eventual *steady state* consists solely of the oscillation of frequency ω. Why should the situation be different in a nonlinear system? The explanation usually offered is as follows: Any oscillation of a nonlinear system contains the higher harmonics in profusion. Hence it is possible that an external force with a frequency the same as one of these might be able to excite and sustain the harmonic of lowest frequency. Of course this requires that the damping (more precisely the ratio c/H) be not too great and that proper precautions of various kinds be taken.

We shall not attempt to present a solution of the problem of subharmonic

[3] For literature, see [2, 3, 17, 18].

response for the Duffing equation in all generality. Rather, we shall treat only one special case. Also, damping will be neglected even though it should be shown that subharmonic response occurs in spite of damping. This could be done, but it would complicate our calculations unnecessarily without causing any significant qualitative changes in the results, particularly if the damping is small. In any case we are interested here, as in our previous discussion, in qualitative rather than accurate quantitative results.

There is some advantage in introducing once more the variable $\theta=\omega t$ as the new independent variable. The differential equation is

$$\omega^2 x'' + (\alpha x + \beta x^3) = F \cos \theta, \tag{27}$$

in which it is to be remembered that ω represents the frequency of the applied force $F \cos \omega t$. Our object is to find a periodic solution with frequency ω/n. We restrict ourselves to the case $n=3$. We apply the Fourier series method and set

$$x = A_{1/3} \cos \frac{\theta}{3} + A_1 \cos \theta + A_{5/3} \cos \frac{5\theta}{3} + \cdots$$

by analogy with the usual or *harmonic* case. (Terms in even multiples of $\theta/3$ and sine terms drop out). Upon substitution of the series in (2) one finds the relations

$$\left(\alpha - \frac{\omega^2}{9}\right) A_{1/3} + \tfrac{3}{4}\beta[A_{1/3}^3 + A_{1/3}^2 A_1 + 2A_{1/3}A_1^2 + \cdots] = 0, \tag{28}$$

$$(\alpha - \omega^2)A_1 + \tfrac{1}{4}\beta[A_{1/3}^3 + 6A_{1/3}^2 A_1 + 3A_1^3 + \cdots] = F. \tag{29}$$

. . . . . .

Equations (28) and (29) take the place of equation (8) which was fundamental for the *harmonic* case.

Again we are faced with a problem of interpretation. Our previous experience offers a guide: we set $\beta=0$ (i.e., we consider the linear case) and observe that $A_{1/3}$ must be taken zero unless $\omega=3\sqrt{\alpha}$. If, however, $\omega=3\sqrt{\alpha}$ then $A_{1/3}$ can be taken arbitrarily, while $A_1=-(1/8\alpha)F$ is determined for a given value of F. The term $A_{1/3} \cos \theta/3$ is evidently the free oscillation of arbitrary amplitude which may be superimposed on the forced vibration $-(F/8\omega) \cos \theta$ in the linear case. Hence we should prescribe $A_{1/3}$ arbitrarily for $\beta \neq 0$ but then hold it fixed. The quantity F should also be prescribed, but the quantities ω and A_1 which were fixed for $\beta=0$ should now be considered as functions of $A_{1/3}$ and F.

With this in mind we rewrite equations (28) and (29) in slightly different form in order to perform iterations conveniently. This yields, after a division of (28) by $A_{1/3} \neq 0$:

$$\omega^2 = 9\alpha + \tfrac{27}{4}\beta[A_{1/3}^2 + A_{1/3}A_1 + 2A_1^2 + \cdots], \tag{30}$$
$$-8\alpha A_1 = F + (\omega^2 - 9\alpha)A_1 - \tfrac{1}{4}\beta[A_{1/3}^3 + 6A_{1/3}^2A_1 + 3A_1^3 + \cdots],$$

or

$$-8\alpha A_1 = F - \tfrac{1}{4}\beta[A_{1/3}^3 - 21A_{1/3}^2A_1 - 27A_{1/3}A_1^2 - 51A_1^3 + \cdots], \tag{31}$$

the last equation resulting through elimination of ω^2.

We begin the iterations with the values for $\beta = 0$, i.e., with $A_{1/3}$ prescribed, $\omega = 3\sqrt{\alpha}$, and $A_1 = -F/8\alpha = A$. The next step yields

$$\omega^2 = 9\alpha + \tfrac{27}{4}\beta[A_{1/3}^2 + A_{1/3}A + 2A^2], \tag{32}$$

$$A_1 = A + \tfrac{1}{32}\beta[A_{1/3}^3 - 21A_{1/3}^2A - 27A_{1/3}A^2 - 51A^3]. \tag{33}$$

Equation (32) is readily discussed: it represents an ellipse or a hyperbola in an ω-$A_{1/3}$-plane, depending on the sign of β. Also ω has an extremum for $A_{1/3} = -\frac{1}{2}A$ and ω^2 has as value there

$$\omega^2 = 9(\alpha + \tfrac{21}{16}\beta A^2). \tag{34}$$

Thus the subharmonic vibration can exist only if

$$\omega \lesseqgtr 3\sqrt{\alpha + \tfrac{21}{16}\beta A^2} \quad \text{for} \quad \begin{matrix} \beta < 0 \\ \beta > 0. \end{matrix} \tag{35}$$

If $\beta \neq 0$, we may conclude that no vibration with $\omega = 3\sqrt{\alpha}$ can exist; i.e., no subharmonic response with exactly the frequency of the linearized problem can exist. Some authors describe subharmonic response as an oscillation with exactly the frequency of the linearized problem excited by a force with three times this frequency. At least in the case of the Duffing equation, such an oscillation cannot occur.

Interesting conclusions can also be drawn from relation (33), which determines the second Fourier coefficient of the subharmonic vibration as a function of $A_{1/3}$. For $A_{1/3} = 0$ relation (28) is satisfied identically while relation (29) reduces to relation (8) for the "harmonic" case. Hence the subharmonic vibration results *through bifurcation from the harmonic one.* This takes place for

$$A_1 = A - \tfrac{51}{32}\beta A^3. \tag{36}$$

In addition we find for $A_{1/3} = 0$

$$dA_1/d(\omega^2) = -\frac{A}{8} > 0 \tag{37}$$

and this determines the direction in which the new branch leaves the original one.

Fig. 8 indicates the nature of the curves for $A_{1/3}$ and $|A_1|$ for both a hard and a soft spring.

Our calculations are accurate only for values of $A_{1/3}$ that are small in comparison with those for A_1, i.e., in the neighborhood of the bifurcation point. More interesting from the point of view of the applications are the solutions for which $A_{1/3}$ is large in comparison with A_1, i.e., those for which the subharmonic component dominates. It is very plausible that these solutions will, like their *harmonic* counterparts, lie in the neighborhood of those for the free oscillation if the force amplitude F is small. Of course, the response curve for the free oscillation in question would be the one beginning at $\omega=3\sqrt{\alpha}$ and not that beginning at $\omega=\sqrt{\alpha}$.

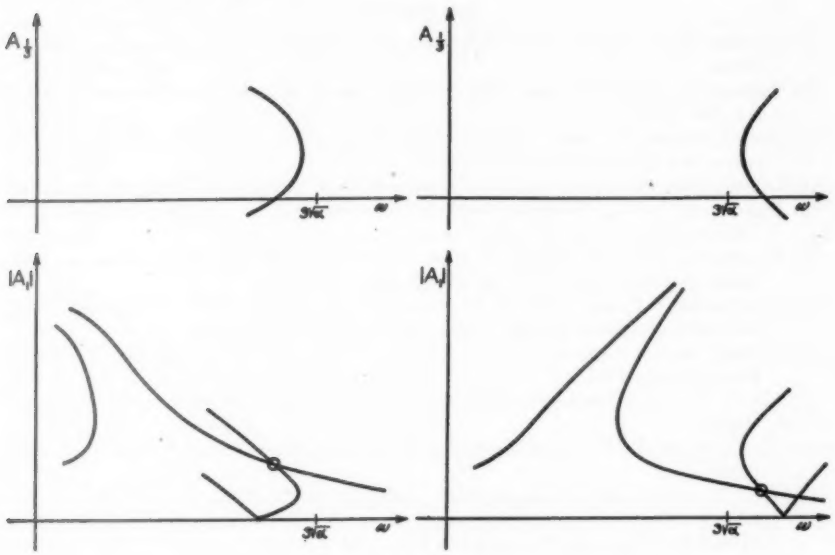

FIG. 8

If the subharmonic component dominates sufficiently so that the resultant solution $x(\theta)$ is monotone in a half period from the maximum to the minimum, then it is possible to apply the method of Rauscher to obtain it.

There are cases which are extreme in another sense, i.e., those in which the dominating harmonic component has a frequency that is a very small fraction (instead of 1/3) of the impressed frequency. In cases involving nonlinear damping such phenomena occur—subharmonic oscillations up to the 200th have been observed, apparently.

9. Other periodic solutions. The harmonic and subharmonic oscillations so far considered do not exhaust the possibilities of periodic solutions. There exists a great variety of other periodic solutions which arise near $\omega=\sqrt{\alpha}/n$, $n=2, 3, \cdots$, in contrast with the subharmonic oscillations which arise near

$\omega = n\sqrt{\alpha}$. Among them are periodic solutions in which the harmonic component of frequency $n\omega$ is dominant. Since the fundamental frequency is ω in such oscillations, they are not exact counterparts of the subharmonic ones; nevertheless they might be called superharmonic oscillations. They can be obtained by a perturbation method beginning with the solution of the linearized problem ($\beta = 0$) $x = A_1 \cos \omega t + A_n \cos n\omega t$ for $\omega = \sqrt{\alpha}/n$, $A_1 = n^2/(n^2-1)F$, while A_n is prescribed arbitrarily. Such solutions have been investigated theoretically, cf. [10, 12, 13], by a different method. They do not appear to have been observed experimentally.

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ON THE DEFLECTION OF ANISOTROPIC THIN PLATES*

BY

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1. Introduction. The elastic medium considered in this paper[1] is assumed to have at each point at least one plane of elastic symmetry parallel to the neutral plane of the plate, which is taken as the xy plane. The theory of bending of thin plates possessing this type of anisotropy has been developed mainly by Boussinesq,[2] Voigt,[3] and Lechnitzky.[4] It is based on the usual assumptions of thin plate theory, leading to the following relations which are valid throughout the (small) thickness $2h$ of the plate

$$\tau_{zz} = 0, \tag{1.1}$$

$$u = -z\frac{\partial w}{\partial x}, \qquad v = -z\frac{\partial w}{\partial y}, \tag{1.2}$$

where w denotes the (small) deflection of the neutral surface of the plate, while u and v are the displacements in the x- and y-directions respectively.

If the equilibrium conditions are used together with *Generalized Hooke's Law*, (1.1), and (1.2), it is possible to express[5] the quantities characterizing the state of stress in terms of the partial derivatives of the deflection w. Then by considerations of equilibrium w is found to satisfy the differential equation:

$$b_{11}\frac{\partial^4 w}{\partial x^4} + 3b_{16}\frac{\partial^4 w}{\partial x^3 \partial y} + 2(b_{12} + b_{66})\frac{\partial^4 w}{\partial x^2 \partial y^2} + 3b_{26}\frac{\partial^4 w}{\partial x \partial y^3} + b_{22}\frac{\partial^4 w}{\partial y^4} = \frac{3q(x, y)}{2h^3}, \tag{1.3}$$

where b_{ij} are constants depending upon the elastic material and $q(x, y)$ is the normal load per unit area on the upper face, the lower face being free.

The problem, therefore, is to find a solution $w(x, y)$ of (1.3) which satisfies

* Received Nov. 18, 1942.

[1] The author wishes to express his appreciation to Professor I. S. Sokolnikoff, who proposed the problem and suggested the manner of approach.

[2] M. J. Boussinesq, Journal de Mathématiques (Liouville) Ser. 3, **5**, 329 (1879).

[3] W. Voigt, *Kompendium der Theoretischen Physik*, Leipzig, 1895, pp. 436–455.

[4] S. G. Lechnitzky, Applied Mathematics and Mechanics, Leningrad, **2**, 181–210, (1938).

[5] An exposition of the theory is given in I. S. Sokolnikoff's *Mathematical Theory of Elasticity*, (mimeographed, Brown University, 1941), pp. 319–329. His notation is adopted in the present discussion.

the prescribed conditions at the boundary (edge) C_0 of the neutral surface of the plate. This general problem has not been solved even in the case of very simple boundaries. Some success was achieved by various writers[6] investigating special shapes of plates made of material with three mutually perpendicular planes of elastic symmetry (orthotropic material). In the present paper a general method of solution is indicated when the boundary C_0 is an analytic curve. Detailed solution, illustrating the procedure, is carried out in the case of a clamped elliptic plate and a polynomial loading function $q(x, y)$. The complicated form of the solutions obtained is due to the generality of the problem. For a given material the values of the parameters b_{ij} in (1.3) are specified and the computations become relatively simple.

2. Preliminary considerations. Utilizing the fact that the potential energy of any realizable state of stress is always nonnegative, Lechnitzky[7] proved that the roots μ_k of the characteristic equation

$$b_{11} + 3b_{16}\mu + 2(b_{12} + b_{66})\mu^2 + 3b_{26}\mu^3 + b_{22}\mu^4 = 0 \tag{2.1}$$

corresponding to (1.3), must be conjugate complex numbers, say,

$$\begin{aligned} &\mu_1 = \alpha_1 + i\beta_1; \qquad \mu_3 = \bar{\mu}_1 = \alpha_1 - i\beta_1; \qquad \beta_1 \neq 0; \\ &\mu_2 = \alpha_2 + i\beta_2; \qquad \mu_4 = \bar{\mu}_2 = \alpha_2 - i\beta_2; \qquad \beta_2 \neq 0. \end{aligned} \tag{2.2}$$

Consider the complex variables z_1 and z_2, related to x and y as follows:

$$z_k = x + \mu_k y = (x + \alpha_k y) + i(\beta_k y), \qquad k = 1, 2. \tag{2.3}$$

This formula specializes to z_0, the complex variable in the original xy plane when $\mu_0 = i$. The relationship between the complex plane of z_0 and that of z_1 or z_2 is

$$z_k = p_k z_0 + \bar{q}_k \bar{z}_0, \qquad k = 1, 2, \tag{2.4}$$

where

$$p_k = \tfrac{1}{2}(1 - i\mu_k) \quad \text{and} \quad q_k = \tfrac{1}{2}(1 - i\bar{\mu}_k). \tag{2.4a}$$

Geometrically, (2.4) corresponds to affine transformations of the z_0 plane. Since $\beta_k \neq 0$, these transformations possess inverses, given by:

$$z_0 = \frac{1}{\beta_k}(\bar{p}_k z_k - \bar{q}_k \bar{z}_k), \qquad k = 1, 2. \tag{2.5}$$

Let w_0 designate a particular integral of (1.3). Lechnitzky[8] has shown that the most general solution of (1.3) can be expressed in the form:

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[6] For instance, M. T. Huber, *Probleme der Statik technisch wichtiger orthotroper Platten*, Warszawa, 1929; W. McDaniels, thesis, University of Wisconsin, 1940; S. G. Lechnitzky, *loc. cit.*, under assumption of vanishing normal load $q(x, y)$.

[7] S. G. Lechnitzky, *loc. cit.* [8] S. G. Lechnitzky, *loc. cit.*

$$w = f_1(z_1) + \bar{f}_1(\bar{z}_1) + f_2(z_2) + \bar{f}_2(\bar{z}_2) + w_0, \tag{2.6}$$

where $f_k(z_k)$ are arbitrary functions of z_k, and $\bar{f}(\bar{z}_k)$ are their conjugates. These functions are analytic in the regions which correspond through (2.4) to the region occupied by the plate. If the roots μ_1 and μ_2 in (2.2) are not distinct, the solution w assumes the form

$$w = \bar{z}_1 f_1(z_1) + z_1 \bar{f}_1(\bar{z}_1) + g_1(z_1) + \bar{g}_1(\bar{z}_1) + w_0. \tag{2.7}$$

When the region in the xy plane is simply-connected, these analytic functions are also single-valued. The present discussion will be confined to simply-connected regions bounded by analytic curves C_0. Let the parametric representation of the boundary be:

$$C_0: \quad z_0 = x + iy = F_1(t) + iF_2(t), \qquad 0 \leqq t < 2\pi. \tag{2.8}$$

The period of the functions $F_s(t)$ is 2π; their Fourier coefficients are

$$d_{ns} = \frac{1}{\pi}\int_0^{2\pi} F_s(t) \cos nt\, dt; \qquad e_{ns} = \frac{1}{\pi}\int_0^{2\pi} F_s(t) \sin nt\, dt. \tag{2.8a}$$

The boundaries of the corresponding regions in z_k planes, by virtue of (2.3), are

$$C_k: \quad z_k = F_1(t) + \mu_k F_2(t), \qquad k = 1, 2. \tag{2.9}$$

It should be noted that the homogeneous deformations (2.4) and hence the contours C_k are determined at the outset by the anisotropy of the plate. The controlling parameters are five in number, namely b_{11}, b_{16}, $(b_{12}+b_{66})$, b_{26}, b_{22}, or b_{11}, α_1, β_1, α_2, β_2.

3. Outline of procedure. The form of the solution (2.6) suggests that some simplification could be achieved by considering the boundary conditions directly in the z_k planes. The boundary conditions usually consist of two functional relations between the deflection w and its partial derivatives of at most third order on C_0

$$G_n\left(w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \cdots, \frac{\partial^3 w}{\partial y^3}\right) = 0, \qquad n = 1, 2. \tag{3.1}$$

If one regards z_k and $\bar{z}_k$ as the independent variables, then

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z_k} + \frac{\partial}{\partial \bar{z}_k}; \qquad \frac{\partial}{\partial y} = \mu_k \frac{\partial}{\partial z_k} + \bar{\mu}_k \frac{\partial}{\partial \bar{z}_k}. \tag{3.2}$$

Hence, the boundary conditions (3.1) can be formulated along either C_1 or C_2:

$$G_n^k\left(w, \frac{\partial w}{\partial z_k}, \frac{\partial w}{\partial \bar{z}_k}, \cdots, \frac{\partial^3 w}{\partial \bar{z}_k^3}\right) = 0, \qquad n, k = 1, 2. \tag{3.3}$$

It is desirable to express the boundary conditions in terms of a single variable. To effect this, suitably chosen neighborhoods of contours C_k in z_k planes are mapped conformally into some neighborhoods of circumferences γ_k of unit radius in new complex ζ_k planes in such a way that points on C_k correspond to points on γ_k.[9] Let φ_k represent the polar angle in the ζ_k planes. Identical values of φ_1 and φ_2, $0 \leqq \varphi_2 < 2\pi$, can be made to correspond to one and the same value of the parameter t on C_0, $0 \leqq t < 2\pi$, by the proper choice of the mapping functions

$$z_k = \omega_k(\zeta_k), \qquad k = 1, 2. \tag{3.4}$$

Laurent expansions for $\omega_k(\zeta_k)$ are found in terms of the Fourier coefficients of $F_s(t)$. Let the common value of ζ_1 and ζ_2 on γ_k be denoted by $\sigma \equiv e^{it}$; then by virtue of the relations

$$\frac{\partial}{\partial z_k} = \frac{1}{\omega_k'(\zeta_k)} \frac{\partial}{\partial \zeta_k}; \qquad \frac{\partial}{\partial \bar{z}_k} = \frac{1}{\bar{\omega}_k'(\bar{\zeta}_k)} \frac{\partial}{\partial \bar{\zeta}_k}, \tag{3.5}$$

the boundary conditions (3.2) can be expressed in terms of σ alone:

$$\Gamma_n\left(w, \frac{\partial w}{\partial \sigma}, \frac{\partial w}{\partial \bar{\sigma}}, \cdots, \frac{\partial^2 w}{\partial \bar{\sigma}^2}\right) = 0, \qquad n = 1, 2. \tag{3.6}$$

The deflection w may have the forms

$$w = \phi_1(\zeta_1) + \bar{\phi}_1(\bar{\zeta}_1) + \phi_2(\zeta_2) + \bar{\phi}_2(\bar{\zeta}_2) + w_0, \tag{3.7}$$

$$w = \bar{\omega}_1(\bar{\zeta}_1)\phi_1(\zeta_1) + \omega_1(\zeta_1)\bar{\phi}_1(\bar{\zeta}_1) + \psi_1(\zeta_1) + \bar{\psi}_1(\bar{\zeta}_1) + w_0, \tag{3.8}$$

where

$$\phi_k(\zeta_k) \equiv f_k\{\omega_k(\zeta_k)\} \quad \text{and} \quad \psi_k(\zeta_k) \equiv g_k\{\omega_k(\zeta_k)\}$$

are undetermined functions, analytic and single-valued in some neighborhoods of γ_k. Accordingly one has

$$\phi_k(\zeta_k) = \sum_{m=-\infty}^{+\infty} A_{mk}\zeta_k^m; \qquad \psi_k(\zeta_k) = \sum_{m=-\infty}^{+\infty} B_{mk}\zeta_k^m. \tag{3.9}$$

Expanding Γ_n in (3.6) into powers of σ and equating coefficients of like powers to zero, one may expect to obtain recursion formulae for the determination of the coefficients A_{mk} and B_{mk} in (3.9). This, in essence, solves the problem.

4. The mapping functions. The mapping functions $\omega_k(\zeta_k)$, if they exist, must be analytic and one-valued at least in narrow rings around the circumferences γ_k where they can be represented by convergent Laurent series, say

$$z_k = \sum_{m=-\infty}^{+\infty} D_{km}\zeta_k^m, \qquad k = 1, 2. \tag{4.1}$$

[9] This will be recognized as an extension of the scheme of N. I. Mushelisvili. See Mathematische Annalen, **107**, 282–312, (1932).

The correspondence between points on the curves C_k and γ_k is expressed through the equation

$$C_k: \quad z_k = \sum_{m=-\infty}^{+\infty} D_{km}\sigma_k^m = \sum_{m=-\infty}^{+\infty} D_{km}(\cos m\varphi_k + i \sin m\varphi_k). \tag{4.2}$$

The parametric representations of C_k are:

$$C_k: z_k = \tfrac{1}{2}(d_{01} + \mu_k d_{02}) + \sum_{n=1}^{\infty} \{(d_{n1} + \mu_k d_{n2}) \cos nt + (e_{n1} + \mu_k e_{n2}) \sin nt\}. \tag{4.3}$$

Letting identical values of φ_1 and φ_2 correspond to the same value of the parameter t, $\varphi_k = t$, one can solve for the coefficients D_{km} from (4.2) and (4.3):

$$\begin{aligned} D_{kn} &= \tfrac{1}{2}\{(d_{n1} + \alpha_k d_{n2} + \beta_k e_{n2}) - i(e_{n1} + \alpha_k e_{n2} - \beta_k d_{n2})\}, \\ D_{k,-n} &= \tfrac{1}{2}\{(d_{n1} + \alpha_k d_{n2} - \beta_k e_{n2}) + i(e_{n1} + \alpha_k e_{n2} + \beta_k d_{n2})\}, \quad n \geqq 0. \end{aligned} \tag{4.4}$$

It will be shown presently that

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|D_{kn}|} < 1, \qquad \overline{\lim_{n\to\infty}} \sqrt[n]{|D_{k,-n}|} < 1, \tag{4.5}$$

so that series (4.1) with coefficients (4.4) actually converge in some rings around γ_k and the foregoing formal steps are justified. The proof rests on the following theorem:[10]

The necessary and sufficient condition that the periodic function $f(z)$ defined by the series

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nz + b_n \sin nz)$$

be analytic in some strip parallel to and containing the real axis is

$$\overline{\lim_{n\to\infty}} \sqrt[n]{(a_n^2 + b_n^2)} < 1 \quad \text{or} \quad \overline{\lim_{n\to\infty}} \sqrt[n]{|a_n + ib_n|} < 1. \tag{4.6}$$

The functions $F_s(t) = \frac{1}{2}d_{0s} + \sum_{n=1}^{\infty}(d_{ns} \cos nt + e_{ns} \sin nt)$, $s = 1, 2$, are by hypothesis (real) analytic functions. Hence the functions defined by the series

$$\begin{aligned} &\frac{1}{2}\sum_{n=1}^{\infty} \{(d_{n1} + \alpha_k d_{n2} + \beta_k e_{n2}) \cos nt - (e_{n1} + \alpha_k e_{n2} - \beta_k d_{n2}) \sin nt\}, \\ &\frac{1}{2}\sum_{n=1}^{\infty} \{(d_{n1} + \alpha_k d_{n2} - \beta_k e_{n2}) \cos nt + (e_{n1} + \alpha_k e_{n2} + \beta_k d_{n2}) \sin nt\}, \end{aligned} \tag{4.7}$$

are also (real) analytic. Replacing t by the complex variable z which reduces to t on the real axis, one is led to complex functions, satisfying the conditions

[10] This is a slight modification of a theorem found on pp. 125–127 of De La Vallée Poussin's, *On the approximation of functions of a real variable and on quasi-analytic functions*, Rice Institute pamphlet, 1925.

of the preceding theorem. But since (4.6) applied to the functions (4.7) implies (4.5), the existence of regions of convergence of (4.1) containing γ_k is shown.

These regions can be narrowed to rings about[11] γ_k in which the derivatives of the mapping functions have no zeros so that the functions themselves are simple (*schlicht*). The representations of the narrowed regions on the corresponding bands about C_k are therefore one to one and conformal as desired. The foregoing discussion can be extended to any number of planes z_k. If, for instance, one sets $\mu_0 = i$, one obtains a function $\omega_0(\zeta_0)$ mapping conformally the original curve C_0 together with a surrounding band into γ_0, the circumference of unit radius in a new ζ_0 plane, and its neighborhood.

5. Mapping of elliptic regions. The procedure described in the preceding section constitutes a practical scheme for the determination of the mapping functions $\omega_k(\zeta_k)$ once the shape of the plate is known. In the case of the elliptic boundary

$$C_0: \quad z_0 = a \cos t + ib \sin t, \tag{5.1}$$

formulae (4.1) and (4.4) yield the mapping function

$$z_k = \omega_k(\zeta_k) = s_k\left(\zeta_k + \frac{R_k}{\zeta_k}\right), \tag{5.2}$$

where

$$R_k = \frac{r_k}{s_k}; \qquad r_k = \tfrac{1}{2}(a + i\mu_k b); \qquad s_k = \tfrac{1}{2}(a - i\mu_k b).$$

It follows from elementary considerations that the deformed contours in z_k planes

$$C_k: \quad z_k = a \cos \varphi + \mu_k b \sin \varphi \tag{5.3}$$

are ellipses whose major axes make angles δ_k with the real x_k axes, where

$$\tan (2\delta_k) = \frac{2\alpha_k\beta_k b^2}{a^2 + b^2(\alpha_k^2 - \beta_k^2)} = \operatorname{am} \{4r_k s_k\}. \tag{5.4}$$

It also follows from (5.2) that the families of circles concentric with γ_k in the ζ_k planes correspond to the families of confocal ellipses with foci at $2\sqrt{r_k s_k}$ in the z_k planes. In particular, to the circumferences γ_k correspond the elliptical contours (5.3) and to the smaller[12] circumferences with radii $\sqrt{|R_k|}$, the degenerate ellipses of the families, i.e., the double segments joining the foci. These degenerate ellipses can be represented in the form

[11] γ_k may well be the outer boundary of these annuli when the derivatives $F_s'(t)$, $s = 1, 2$, vanish simultaneously. Thus the class of curves to which this discussion is applicable is somewhat larger than the class of analytic functions as usually defined.

[12] If μ_1 and μ_2 are defined to be the roots whose imaginary parts are positive, $\beta_k > 0$, then $\sqrt{|R_k|} < 1$.

$$z_k = 2\sqrt{(r_k s_k)}\cos(\varphi - \delta_k') \tag{5.5}$$

where δ_k' are the amplitudes of $\sqrt{R_k}$, the ζ_k plane images of the foci $2\sqrt{r_k s_k}$. Since the derivatives $\omega_k'(\zeta_k)$ vanish only at $\sqrt{R_k}$, one may choose for the rings in which the mapping functions are to be *schlicht* the annuli $1 \leqq \zeta_k < \sqrt{|R_k|}$. The corresponding regions in z_k planes comprise the interiors of the ellipses C_k made doubly-connected by the introduction of slits (5.5) between the two foci.

From (5.5) it is clear that every point z_k on the slit corresponds to two distinct points, $\sqrt{|R_k|}e^{i\varphi}$ and $\sqrt{|R_k|}e^{i(2\delta_k'-\varphi)}$, on the inner boundary of the above ring. Since the analytic functions $f_k(z_k)$ are single valued inside C_k, the transformed functions $\phi_k(\zeta_k)$ must assume identical values at the two points. Hence,

$$\sum_{m=-\infty}^{+\infty} A_{mk}\sqrt{|R_k|^m}\, e^{im\varphi} = \sum_{m=-\infty}^{+\infty} A_{mk}\sqrt{|R_k|^m}\, e^{im(2\delta_k'-\varphi)}, \tag{5.6}$$

so that

$$A_{-n,k} = R_k^n A_{nk}, \qquad n > 0,$$

and finally

$$\phi_k(\zeta_k) = \sum_{n=0}^{\infty} A_{nk}'\left(\zeta_k^n + \frac{R_k^n}{\zeta_k^n}\right); \qquad \psi_k(\zeta_k) = \sum_{n=0}^{\infty} B_{nk}'\left(\zeta_k^n + \frac{R_k^n}{\zeta_k^n}\right). \tag{5.7}$$

The question arises whether the restriction (5.6) is sufficient to insure that the functions $F_k(z_k)$ obtained by the inverse transformation

$$\zeta_k = \frac{1}{2s_k}\left\{z_k + \sqrt{(z_k^2 - 4r_k s_k)}\right\} \tag{5.8}$$

from the functions $\phi_k(\zeta_k)$ in (5.7) are single-valued in the full, simply-connected interiors of the ellipses C_k. An affirmative answer is reached after application of *Schwarz's Reflection Principle* to the potentially different branches of $F_k(z_k)$ on the opposite sides of the slits. Therefore, if the unknown coefficients in (5.7) are determined from similarly transformed boundary conditions, the resulting functions will correspond to the solutions of the original problem.

6. Uniqueness of solutions. If in (2.7) the functions $f_1(z_1)$ and $g_1(z_1)$ were replaced by $f_1(z_1)+f_1^*(z_1)$ and $g_1(z_1)+g_1^*(z_1)$ where

$$Re\left\{\bar{z}_1 f_1^*(z_1) + g_1^*(z_1)\right\} = 0, \tag{6.1}$$

the value of w would remain unaltered. The extent of arbitrariness in the choice of $f_1(z_1)$ and $g_1(z_1)$ is determined next by finding the most general analytic functions $f_1^*(z_1) = u_1 + iv_1$ and $g_1^*(z_1) = u_2 + iv_2$ satisfying (6.1), or the equivalent condition

$$x_1u_1 + y_1v_1 + u_2 = 0. \tag{6.2}$$

Utilizing the fact that u_1, v_1, u_2, and v_2 are harmonic functions, conjugate in pairs, one readily finds from (6.2) that

$$\frac{\partial u_1}{\partial x_1} = 0, \qquad \frac{\partial v_1}{\partial y_1} = 0,$$

and therefore

$$f_1^*(z_1) = \lambda_1 i z_1 + (\lambda_2 + i\lambda_3), \tag{6.3a}$$

where λ_j are arbitrary real constants. Substituting this expression into (6.1) one arrives at

$$g_1^*(z_1) = -(\lambda_2 - i\lambda_3)z_1 + i\lambda_4. \tag{6.3b}$$

The foregoing functions are transformed by (5.2) into

$$\begin{aligned} \phi_1^*(\zeta_1) &= \lambda_1 i s_1\left(\zeta_1 + \frac{R_1}{\zeta_1}\right) + \lambda_2 + i\lambda_3; \\ \psi_1^*(\zeta_1) &= -(\lambda_2 - i\lambda_3)s_1\left(\zeta_1 + \frac{R_1}{\zeta_1}\right) + i\lambda_4. \end{aligned} \tag{6.4}$$

It is desirable to eliminate the arbitrariness inherent in the form of solutions by choosing the parameters λ_j so as to simplify the expansions (5.7) of $\phi_1(\zeta_1)$ and $\psi_1(\zeta_1)$. If

$$\lambda_1 = \frac{-2 \text{ Im } \{A'_{11}\}}{a + b\beta_1}; \qquad \lambda_2 + i\lambda_3 = -2A'_{01}; \qquad \lambda_4 = -2 \text{ Im } \{B'_{01}\};$$

the functions $\phi_1+\phi_1^*$ and $\psi_1+\psi_1^*$ reduce to

$$\sum_{n=1}^{\infty} A_{n1}\left(\zeta_1^n + \frac{R_1^n}{\zeta_1^n}\right) \quad \text{and} \quad \sum_{n=0}^{\infty} B_{n1}\left(\zeta_1^n + \frac{R_1^n}{\zeta_1^n}\right), \tag{6.5}$$

with $A_{11} = \overline{A}_{11}$ and $B_{01} = \overline{B}_{01}$. That the solutions of the form (6.5) are uniquely determined by the boundary conditions is shown in Section 8.

A similar argument shows that one needs to consider only the functions

$$\phi_1(\zeta_1) = \sum_{n=0}^{\infty} A_{n1}\left(\zeta_1^n + \frac{R_1^n}{\zeta_1^n}\right); \qquad \phi_2(\zeta_2) = \sum_{n=2}^{\infty} A_{n2}\left(\zeta_2^n + \frac{R_2^n}{\zeta_2^n}\right), \tag{6.6}$$

with $A_{01} = \overline{A}_{01}$ and $A_{22} = \overline{A}_{22}$, when the roots μ_1 and μ_2 are distinct.

7. Boundary conditions. Let θ designate the angle between the outward normal to the boundary C_0 and x axis. Then,

$$\sin\theta = -\frac{1}{2ds}(dz_0 + d\bar{z}_0) = -\frac{1}{2ds\beta_k}(i\bar{\mu}_k dz_k - i\mu_k d\bar{z}_k),$$
$$\cos\theta = \frac{i}{2ds}(dz_0 - d\bar{z}_0) = \frac{i}{2ds\beta_k}(dz_k - d\bar{z}_k), \tag{7.1}$$

where dz_0 and ds are differentials along C_0 and dz_k the corresponding (dependent) differentials in the z_k planes. The equations (7.1) and (3.2), together with the mapping functions (3.4) make it possible to formulate the boundary conditions in terms of σ on the circumferences γ_k. In the case of a clamped plate one has the conditions:

$$w(x, y) = 0, \qquad \frac{\partial w}{\partial n} = \left(\cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}\right)w(x, y) = 0, \tag{7.2}$$

on C_0, which are equivalent to the vanishing, on γ_k, of the expressions[13]

$$w(\sigma, \bar{\sigma}) = 0, \tag{7.3a}$$

$$-\frac{i}{\beta_k\sigma}\frac{d\sigma}{ds}\left[\left\{\sigma(p_k\bar{p}_k + q_k\bar{q}_k) + \frac{1}{\sigma}\frac{\bar{\omega}_k'(\bar{\sigma})}{\omega_k'(\sigma)}2p_k\bar{q}_k\right\}\frac{\partial}{\partial\sigma}\right.$$
$$\left. + \left\{\sigma\frac{\omega_k'(\sigma)}{\bar{\omega}_k'(\bar{\sigma})}2\bar{p}_k q_k + \frac{1}{\sigma}(p_k\bar{p}_k + q_k\bar{q}_k)\right\}\frac{\partial}{\partial\bar{\sigma}}\right]w(\sigma, \bar{\sigma}) = 0, \tag{7.3b}$$

where $d\sigma$ is dependent on dz_0. Obviously, to the part of $w(\sigma, \bar{\sigma})$ obtained by transformation of $f_i(z_i)$ and $g_i(z_i)$ one applies the operator in (7.3b) with $k = i$. It is expedient to make use of the mapping function $z_0 = \omega_0(\zeta_0)$ for the transformation of the particular integral w_0. When $k = 0$ the operator in (7.3b) reduces to

$$-\frac{i}{\sigma}\frac{d\sigma}{ds}\left(\sigma\frac{\partial}{\partial\sigma} + \frac{1}{\sigma}\frac{\partial}{\partial\bar{\sigma}}\right)w_0(\sigma, \bar{\sigma}), \tag{7.4}$$

and (5.2) has the form

$$z_0 = \omega_0(\zeta_0) = \tfrac{1}{2}(a - b)\left(k^2\zeta_0 + \frac{1}{\zeta_0}\right), \qquad k^2 = \frac{a + b}{a - b}. \tag{7.5}$$

If the normal load $q(x, y)$ in (1.3) is expressed as a polynomial[14] in x and y of degree $(n_1 - 4)$, w_0 is a polynomial of degree n_1. For instance, when the load is linear

$$q(x, y) = \frac{2h^3}{3}(c_0 + c_1 x + c_2 y), \tag{7.6}$$

where c_i are real parameters, one can take for w_0,

[13] It is noted that, as a consequence of (3.2) and (3.4), σ and $\bar{\sigma}$ are treated as independent variables in the partial differentiation process, whereas the result can be simplified by $\bar{\sigma} = 1/\sigma$, an identity on γ_k.

[14] This may be considered as an approximation to more complicated loading functions.

$$w_0(x, y) = \frac{c_0}{4!\, b_{11}} x^4 + \frac{c_1}{5!\, b_{11}} x^5 + \frac{c_2}{5!\, b_{22}} y^5. \tag{7.7}$$

The following abbreviations referring to values on the original curve C_0 but expressed on γ_k are introduced:

$$\begin{aligned} w_0 &= -K(\sigma) = -\sum_0^{n_1}\left(b_m\sigma^m + \bar{b}_m\frac{1}{\sigma^m}\right), \\ i\sigma\frac{ds}{d\sigma}\frac{\partial w_0}{\partial n} &= -L(\sigma) = -\sum_0^{n_1}\left(a_m\sigma^m + \bar{a}_m\frac{1}{\sigma^m}\right). \end{aligned} \tag{7.8}$$

One readily finds the coefficients a_m and b_m for the special case of the linear load applied to an elliptic plate:

$$\begin{aligned} &a_4 = -\frac{4ba^3c_0}{2^4\,4!\,b_{11}}; \qquad a_5 = -\frac{5ab}{2^5\,5!}\left(\frac{c_1a^3}{b_{11}} - i\frac{c_2b^3}{b_{22}}\right), \\ &a_0 = 3a_4, \qquad a_2 = 4a_4; \qquad a_3 = 5\bar{a}_5, \qquad a_1 = 10a_5, \\ &b_{2m} = \frac{a}{4b}a_{2m}; \qquad (b_{2m+1} \pm \bar{b}_{2m+1}) = \frac{a}{5b}(a_{2m+1} \pm \bar{a}_{2m+1}). \end{aligned} \tag{7.9}$$

The preceding transformation of the particular integral w_0 is valid whether the deflection w has the form (2.6) or (2.7). However, the parts of w involving the unknown functions $f_k(z_k)$ and $g_k(z_k)$ have to be treated somewhat differently in the cases of equal and distinct roots. To avoid repetition only the first case is discussed here.

8. The case of equal roots. Substituting[15] (3.8) into the boundary conditions (7.3a) and (7.3b) and simplifying gives:

$$\bar{\omega}\left(\frac{1}{\sigma}\right)\phi(\sigma) + \omega(\sigma)\bar{\phi}\left(\frac{1}{\sigma}\right) + \psi(\sigma) + \bar{\psi}\left(\frac{1}{\sigma}\right) = K(\sigma), \tag{8.1a}$$

$$\begin{aligned} &\frac{1}{\beta}\left[\sigma(p\bar{p} + q\bar{q}) + \frac{1}{\sigma}\frac{\bar{\omega}'\left(\frac{1}{\sigma}\right)}{\omega'(\sigma)}2p\bar{q}\right] \\ &\qquad\cdot\left[\bar{\omega}\left(\frac{1}{\sigma}\right)\phi'(\sigma) + \omega'(\sigma)\bar{\phi}\left(\frac{1}{\sigma}\right) + \psi'(\sigma)\right] \\ &+\frac{1}{\beta}\left[\sigma\frac{\omega'(\sigma)}{\bar{\omega}'\left(\frac{1}{\sigma}\right)}2\bar{p}q + \frac{1}{\sigma}(p\bar{p} + q\bar{q})\right] \\ &\qquad\cdot\left[\omega(\sigma)\bar{\phi}'\left(\frac{1}{\sigma}\right) + \bar{\omega}'\left(\frac{1}{\sigma}\right)\phi(\sigma) + \bar{\psi}'\left(\frac{1}{\sigma}\right)\right] = L(\sigma). \end{aligned} \tag{8.1b}$$

[15] In this section the subscripts are omitted since the distinction between z_1 and z_2 planes is not involved in the discussion.

In order to avoid a system of an infinite number of equations in infinitely many unknowns the following device is used. The condition (8.1b) is replaced by a combination of (8.1b) and (8.1a). The first boundary condition (8.1a) is an identity in σ and will therefore yield an equality upon differentiation with respect to σ. This equation is multiplied by

$$\frac{\sigma}{\beta}\left\{\frac{\sigma^2\omega'(\sigma)}{\bar{\omega}'\left(\frac{1}{\sigma}\right)}2\bar{p} + (p\bar{p}+q\bar{q})\right\}$$

and added to (8.1b). In the resulting identity, one solves for the expression involving the unknown functions and obtains[16]

$$\bar{\omega}\left(\frac{1}{\sigma}\right)\sigma\phi'(\sigma) + \sigma\omega'(\sigma)\bar{\phi}\left(\frac{1}{\sigma}\right) + \sigma\psi'(\sigma)$$

$$= \frac{1}{2}\left\{\frac{p}{\frac{p}{\sigma}\bar{\omega}'\left(\frac{1}{\sigma}\right) + q\sigma\omega'(\sigma)} - \frac{\bar{q}}{\bar{p}\sigma\omega'(\sigma) + \frac{\bar{q}}{\sigma}\bar{\omega}'\left(\frac{1}{\sigma}\right)}\right\}$$

$$\cdot\left\{\frac{L(\sigma)}{\sigma}\bar{\omega}'\left(\frac{1}{\sigma}\right) + \frac{\sigma K'(\sigma)}{\beta}\left(\sigma\omega'(\sigma)2\bar{p}q + \frac{\bar{\omega}'\left(\frac{1}{\sigma}\right)}{\sigma}(p\bar{p}+q\bar{q})\right)\right\}. \quad (8.2)$$

When the plate is elliptic, (8.1a) and (8.2) read:

$$\bar{s}\left(\frac{1}{\sigma}+\overline{R}\sigma\right)\sum_1^\infty A_m\left(\sigma^m+\frac{R^m}{\sigma^m}\right) + s\left(\sigma+\frac{R}{\sigma}\right)\sum_1^\infty \overline{A}_m\left(\frac{1}{\sigma^m}+\overline{R}^m\sigma^m\right)$$

$$+\sum_0^\infty B_m\left(\sigma^m+\frac{R^m}{\sigma^m}\right) + \sum_0^\infty \overline{B}_m\left(\frac{1}{\sigma^m}+\overline{R}^m\sigma^m\right)$$

$$= \sum_0^{n_1}\left(b_m\sigma^m + \bar{b}_m\frac{1}{\sigma^m}\right), \quad (8.3)$$

$$\bar{s}\left(\frac{1}{\sigma}+\overline{R}\sigma\right)\sum_1^\infty mA_m\left(\sigma^m-\frac{R^m}{\sigma^m}\right) + s\left(\sigma-\frac{R}{\sigma}\right)\sum_1^\infty \overline{A}_m\left(\frac{1}{\sigma^m}+\overline{R}^m\sigma^m\right)$$

$$+\sum_0^\infty mB_m\left(\sigma^m-\frac{R^m}{\sigma^m}\right) = \frac{1}{\sigma\beta(a+b)}\left[\frac{p\sigma^2}{1-\sigma^2/k^2} - \frac{\bar{q}}{1-1/k^2\sigma^2}\right]$$

$$\cdot\left[\bar{s}\left(\frac{1}{\sigma}-\overline{R}\sigma\right)\sum_0^{n_1}\left(a_m\sigma^m+\frac{\bar{a}_m}{\sigma^m}\right) + \tfrac{1}{2}\left\{(b-i\bar{\mu}a)\sigma\right.\right.$$

$$\left.\left.+\frac{1}{\sigma}(b+i\bar{\mu}a)\right\}\sum_0^{n_1} m\left(b_m\sigma^m - \bar{b}_m\frac{1}{\sigma^m}\right)\right]. \quad (8.4a)$$

[16] The expressions in the denominators, $1/\beta(\bar{p}\sigma\omega'(\sigma)+\bar{q}/\sigma\ \bar{\omega}'(1/\sigma))$ and $1/\beta(p/\sigma\ \bar{\omega}'(1/\sigma)+q\sigma\omega'(\sigma))$, never vanish, since $p\bar{p}-q\bar{q}=\beta\neq 0$.

For a circular plate $a=b$, $1/k^2$ vanishes, and the last equation is already expanded in series of powers of σ. Otherwise, the expansion of the right hand member is infinite, since

$$\frac{1}{\sigma}\left\{\frac{p\sigma^2}{1-\sigma^2/k^2}-\frac{\bar{q}}{1-1/k^2\sigma^2}\right\}=p\sigma\sum_{l=0}^{\infty}\left(\frac{\sigma}{k}\right)^{2l}-\frac{\bar{q}}{\sigma}\sum_{l=0}^{\infty}\left(\frac{1}{k\sigma}\right)^{2l}. \tag{8.5}$$

Combining the terms $\bar{s}(1/\sigma-\overline{R}\sigma)$ and $\frac{1}{2}((b-i\bar{\mu}a)\sigma+1/\sigma(b+i\bar{\mu}a))$ with the expansion (8.5), one simplifies the right-hand member of (8.4a):

$$\frac{1}{2\beta(a+b)}\left[(a+\mu\bar{\mu}b)\sum_0^{n_1}\left(a_m\sigma^m+\bar{a}_m\frac{1}{\sigma^m}\right)+i(\bar{\mu}a-\mu b)\sum_0^{n_1}m\left(b_m\sigma^m-\bar{b}_m\frac{1}{\sigma^m}\right)\right]$$

$$-\frac{2ab}{\beta(a^2-b^2)}\left[pq\left(\sum_0^{n_1}\left\{(a_m-mb_m)\sigma^m+(\bar{a}_m+m\bar{b}_m)\frac{1}{\sigma^m}\right\}\right)\sum_1^{\infty}\left(\frac{\sigma}{k}\right)^{2l}\right.$$

$$\left.+\bar{p}\bar{q}\left(\sum_0^{n_1}\left\{(a_m+mb_m)\sigma^m+(\bar{a}_m-m\bar{b}_m)\frac{1}{\sigma^m}\right\}\right)\sum_1^{\infty}\left(\frac{1}{\sigma k}\right)^{2l}\right]. \tag{8.4b}$$

Let Q_n denote the coefficient of σ^n in (8.4b). This coefficient is evaluated by means of relations similar to (7.9). It is found that Q_n vanishes for $n>n_1$.

Recalling that $A_0=0$, $A_1=\overline{A}_1$, and $B_0=\overline{B}_0$, one concludes from the relations between the coefficients of σ^0 in (8.4a) and (8.3):

$$A_1=\frac{Q_0}{(1-P)(s+\bar{s})},\qquad B_0=\frac{1}{2}b_0-\frac{1}{4}\frac{1+P}{1-P}Q_0,\ \text{where } P=R\overline{R}. \tag{8.6}$$

The system of equations between the coefficients of σ^n, $n>1$,

$$\begin{aligned}\bar{s}A_{n+1}+sR\overline{R}^{n+1}\overline{A}_{n+1}+B_n+\overline{R}^n\overline{B}_n&=b_n-\bar{s}\overline{R}A_{n-1}-s\overline{R}^{n-1}\overline{A}_{n-1},\\ \bar{s}(n+1)A_{n+1}-sR\overline{R}^{n+1}\overline{A}_{n+1}+nB_n&=Q_n-\bar{s}\overline{R}(n-1)A_{n-1}-s\overline{R}^{n-1}\overline{A}_{n-1},\end{aligned} \tag{8.7}$$

and of the conjugate equations has a non-vanishing determinant:

$$s\bar{s}D_{n+1}(P)=s\bar{s}\left\{1-(n+1)^2P^n+2n(n+2)P^{n+1}-(n+1)^2P^{n+2}+P^{2n+2}\right\},\quad n>1,$$

because $P<1$. The system, therefore, possesses a unique solution given by the expressions:

$$\begin{aligned}A_{n+1}=\frac{1}{\bar{s}D_{n+1}}&[Q_n(1-(n+1)P^n+nP^{n+1})-\overline{Q}_n\overline{R}^n(n-(n+1)P+P^{n+1})\\&-nb_n(1-P^{n+1})+n(n+1)\bar{b}_n\overline{R}^n(1-P)+\bar{s}\overline{R}D_n(P)A_{n-1}\\&+s\overline{R}^{n-1}E_n(P)\overline{A}_{n-1}],\end{aligned} \tag{8.8}$$

$$B_n = \frac{-1}{D_{n+1}} [Q_n\{1-(n+2)P^{n+1}+(n+1)P^{n+2}\}$$
$$-\overline{Q}_n\overline{R}^n\{(n+1)-(n+2)P+P^{n+2}\}-(n+1)b_n(1-P^{n+2})$$
$$+b_n\overline{R}^n\{(n+1)^2-n(n+2)P-P^{n+2}\}$$
$$+s\overline{R}G_{n+1}(P)A_{n-1}+s\overline{R}^{n-1}E_{n+1}(P)\overline{A}_{n-1}], \tag{8.9}$$

where the following abreviations are used:

$$E_n(P)=\{(n-1)-2nP+(n+1)P^2+(n+1)P^n-2nP^{n+1}+(n-1)P^{n+2}\},$$
$$G_{n+1}(P)=\{2-n(n+1)P^{n-1}+(n-1)(n+2)P^n+(n-1)(n+2)P^{n+1}$$
$$-n(n+1)P^{n+2}+2P^{2n+1}\}.$$

One can calculate any number of coefficients in the expansions of $\phi(\zeta)$ and $\psi(\zeta)$ from those already known, A_0 and A_1, and the recurrence formulae (8.8) and (8.9). If (8.8) is rewritten for $n>n_1$ in the form

$$\frac{A_{n+1}}{A_{n-1}}=\frac{1}{D_{n+1}(P)}\left\{\overline{R}D_n(P)+\frac{s}{\bar{s}}\overline{R}^{n+1}E_n(P)\frac{\overline{A}_{n-1}}{A_{n-1}}\right\}, \tag{8.10}$$

One sees that $\lim_{n\to\infty}|A_{n+1}/A_{n-1}|=|\overline{R}|$. Consequently, $\phi(\zeta)$ and $\psi(\zeta)$ converge and represent analytic functions in the ring $|R|^{3/2}<|\zeta|<|R|^{-1/2}$, which extends beyond the ring corresponding to the interior of the ellipse C_0. The expansions are finite if A_{n-1} and A_n vanish for any $n>n_1$. In the case of the linear load one finds $A_5=A_6=0$, $B_5\neq 0$, so that the solution is a polynomial of fifth degree. Specific examples indicate that the solution is a polynomial when the loading function $q(x, y)$ is a polynomial.

9. Conclusion. The discussion in the case of distinct roots is similar to that of Section 8. For details and results the reader is referred to the author's doctoral thesis.[17]

When $\mu_1=\mu_2=i$, the solution reduces to the results for an isotropic clamped elliptic plate bent by uniformly varying load.[18] Another important special case is that of an orthotropic elliptic plate bent by a linear load, for which the solution is new. There are two possibilities obtained by setting either $\mu_1=i\beta_1$, $\mu_2=i\beta_2$, or $\mu_1=\alpha_1+i\beta_1$, $\mu_2=-\alpha_1+i\beta_1$. The calculations in these and other specific cases are comparatively simple. It is clear that by specifying the values of some or all of the parameters characterizing the material

[17] *On the deflection of anisotropic thin plates*, University of Wisconsin, 1942.

[18] See A. E. H. Love, *Treatise on the Mathematical Theory of Elasticity*, Cambridge, 1927, p. 486.

of the plate, its shape, and the type of loading, a considerable simplification is achieved.

The foregoing method can be extended to the cases of (a) different boundary conditions, (b) different shapes of plates. Many of the devices and formulae, such as those of Sections 4, 6, and 7, remain valid. The extension to the interesting case of infinite doubly-connected regions involves little more than the additional feature of determining the character of the multiple-valued functions $f_k(z_k)$ from the fact that the deflection w remains single-valued and continuous.

SYMMETRICAL JOUKOWSKY AIRFOILS IN SHEAR FLOW†

BY

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1. Problem. The usual two-dimensional theory of airfoils assumes a uniform velocity for points far from the airfoil. There are many applications where this condition is not satisfied. For example near the ground there is a large vertical velocity gradient, and therefore a first approximation to the problem can be obtained by assuming a linear velocity distribution. This has, in fact, been done by H. v. Sanden[1] in connection with O. Lilienthal's experiments in natural wind. However, v. Sanden used a numerical method of integrating the differential equation and carried out the calculation only for a wedge-shaped body. Th. von Kármán* suggested to the author to take up this problem again in order to develop a more complete theory. Hence in the

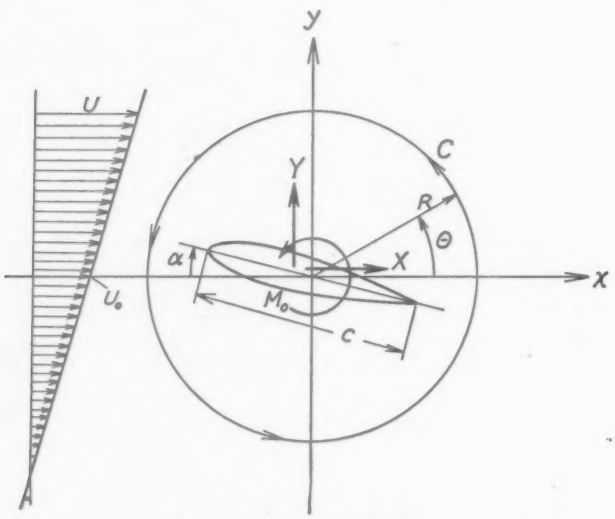

FIG. 1. Body in a Shear Flow.

† Received Dec. 27, 1942.

[1] v. Sanden, H., *Über den Auftrieb im natürlichen Winde*, Zeitschrift f. Math. u. Phys., **61**, 225, (1912).

* The author wishes to thank Dr. von Kármán for suggesting the problem and his kind interest during the course of the work.

first part of this paper, a generalization of the well-known Blasius theorem for calculating aerodynamic forces acting on an airfoil is given. Then the result is applied to the case of symmetrical Joukowsky airfoils and the final data are given in a number of tables and graphs.

2. Method of solution. By setting up the problem as shown in Fig. 1, the velocity distribution far from the airfoil is given by

$$u = U_0\left(1 + K\frac{y}{c}\right) \tag{1}$$

$$v = 0$$

where U_0 is the undisturbed flow velocity along the x-axis and K is the non-dimensional velocity gradient of the undisturbed flow. c is some dimension of the body immersed in the stream, e.g., the chord of the airfoil. Then the vorticity at locations far from the airfoil can be calculated from Eq. (1) as

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -U_0K/c \tag{2}$$

which is a constant. However, in the flow of non-viscous incompressible fluid, the vorticity is associated with the fluid and maintains its strength. Consider the field of flow starting from the far left, where the vorticity is constant and equal to $-U_0K/c$. This value of vorticity is carried with the fluid over the whole field of flow. Therefore, the flow problem on hand is one with constant vorticity distribution.

To satisfy the equation of continuity,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3}$$

the stream function ψ is introduced. It is defined by

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}. \tag{4}$$

u, v are the components of velocity in the x-direction and the y-direction. Due to constant vorticity distribution, the vorticity equation is

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -U_0K/c. \tag{5}$$

By using Eq. (4), Eq. (5) can be written as

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = U_0K/c. \tag{6}$$

Therefore the flow problem is reduced to that of solving Eq. (6).

The stream function ψ_0 of the undisturbed flow given by Eq. (1) is

$$\psi_0 = U_0\left(y + \frac{K}{2}\frac{y^2}{c}\right) \tag{7}$$

which can be easily verified by means of Eq. (4). The mathematical problem can be considerably simplified by introducing the stream function ψ_1 due to the presence of the body defined as

$$\psi = \psi_0 + \psi_1. \tag{8}$$

By substituting this expression for ψ into Eq. (6), the equation for ψ_1 is simply

$$\frac{\partial^2\psi_1}{\partial x^2} + \frac{\partial^2\psi_1}{\partial y^2} = 0. \tag{9}$$

This is the Laplace equation. Therefore any solution of the Laplace equation

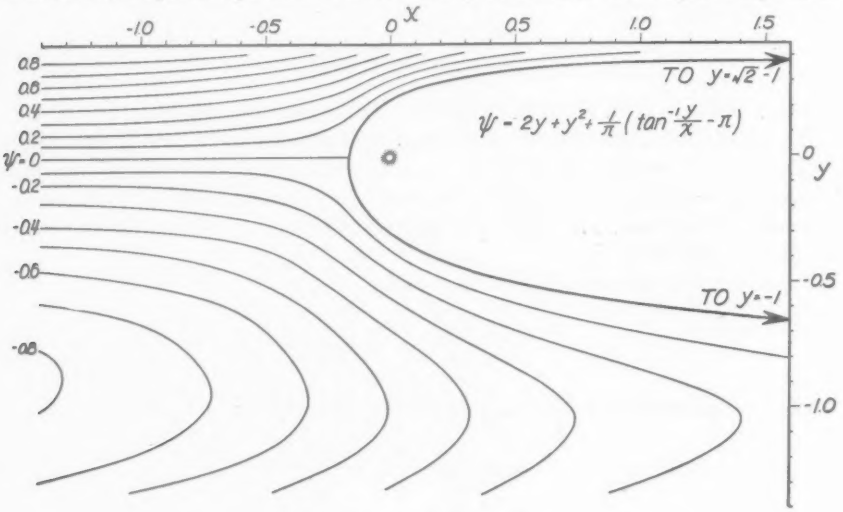

FIG. 2. Source in a Shear Flow.

combined with ψ_0 will satisfy Eq. (6). For example, we can combine ψ_0 with a source

$$\psi_1 = U_0 b_0 \theta \tag{10}$$

or a vortex

$$\psi_1 = U_0 a_0 \log r \tag{11}$$

as shown in Figs. 2, 3, and 4. Here

$$\theta = \tan^{-1}\frac{y}{x}, \qquad r = \sqrt{x^2 + y^2}. \tag{12}$$

It is interesting to notice that in the case of a source the zero stream line, which forms the walls of the "half body" when $K=0$, is no longer symmetrical with respect to the flow direction. The velocity, and hence the pressure, along

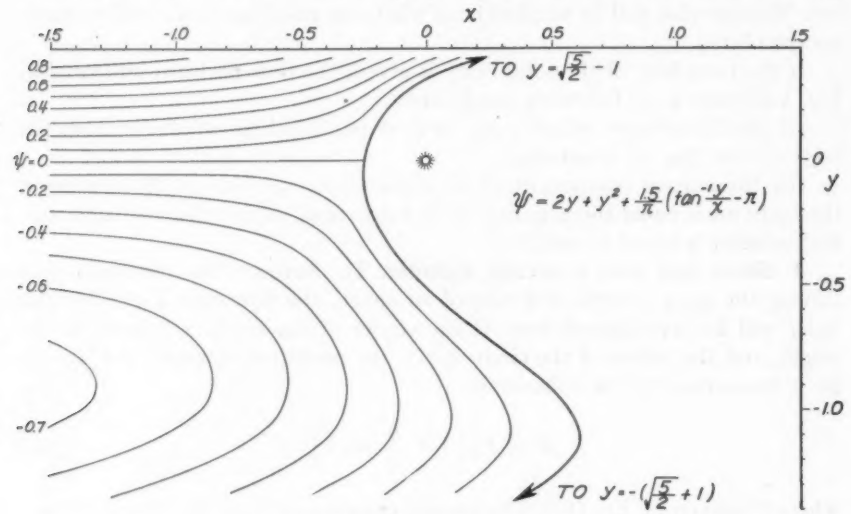

FIG. 3. Source in a Shear Flow.

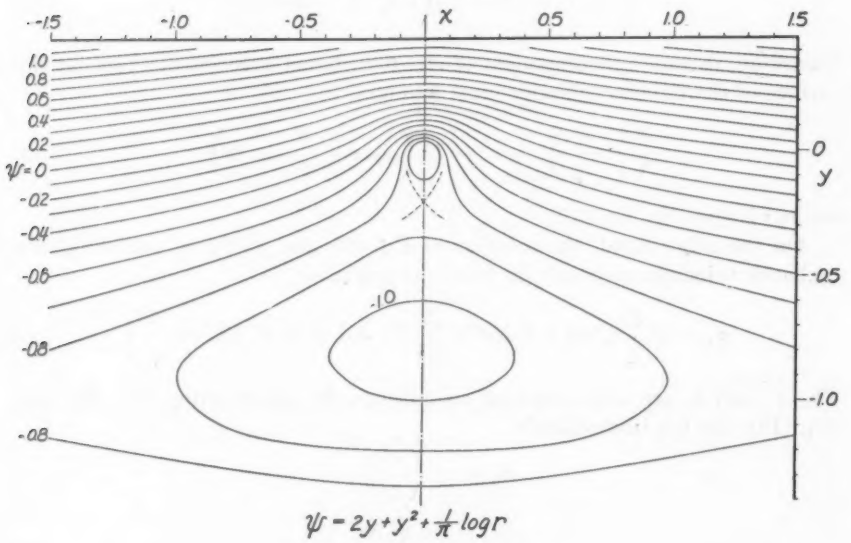

FIG. 4. Vortex in a Shear Flow.

these two branches are also different. If the flow field within the zero stream line is replaced by a solid body, such lack of balance may cause the resultant pressure force to become infinitely large with the infinitely long solid boundary. This surmise will be verified later when the resultant force and moment are calculated.

If the boundary of the solid body is given, the flow problem indicated in Fig. 1 subjects ψ_1 to following conditions:

(i) the disturbance velocity due to ψ_1 must vanish at points far from the body so that Eq. (1) is satisfied;

(ii) the normal component of the disturbance velocity at the surface of the body must equal the negative of that due to ψ_0 so that the resultant normal velocity is equal to zero.

3. Shear flow over a circular cylinder. To illustrate this method before solving the more complicated case of an airfoil, the flow over a circular cylinder will be investigated first. If the center of the circle is located at the origin, and the radius of the circle is $c/2$, the undisturbed velocity U due to ψ_0 at the surface of the cylinder is

$$U = U_0\left(1 + \frac{K}{2}\sin\theta\right) \tag{13}$$

where θ is given by Eq. (12). The normal component U_r of the velocity U is

$$U_r = U\cos\theta = U_0\left(\cos\theta + \frac{K}{4}\sin 2\theta\right). \tag{14}$$

Therefore, the normal component of the disturbance velocity due to ψ_1 at the surface of the circular cylinder must be equal to $-U_r$, or

$$\left(\frac{1}{r}\frac{\partial\psi_1}{\partial\theta}\right)_{r=c/2} = -U_0\left(\cos\theta + \frac{K}{4}\sin 2\theta\right) \tag{15}$$

where r is given by Eq. (12).

On the other hand, the solution of Eq. (9) that will give vanishing disturbance velocities at points far from the origin is

$$\psi_1 = U_0\left[a_0\log r + b_0\theta + \sum_{n=1}^{\infty}(a_n\cos n\theta + b_n\sin n\theta)\frac{1}{r^n}\right]. \tag{16}$$

The a_n and b_n are undetermined coefficients. By substituting Eq. (16) into Eq. (15), one has immediately

$$b_1 = -\left(\frac{c}{2}\right)^2$$

$$a_2 = \left(\frac{c}{2}\right)^2\frac{K}{8}. \tag{17}$$

All the other coefficients vanish. Therefore, the resultant stream function is

$$\psi = \psi_0 + \psi_1 = U_0\left[\left(r - \frac{c^2}{4r}\right)\sin\theta + \frac{K}{2}\left(\frac{r^2}{c}\sin^2\theta + \frac{c^3}{32r^2}\cos 2\theta\right)\right]. \quad (18)$$

At the surface of the cylinder, $r = c/2$, Eq. (18) reduces to

$$\psi = U_0\frac{Kc}{16} \quad (19)$$

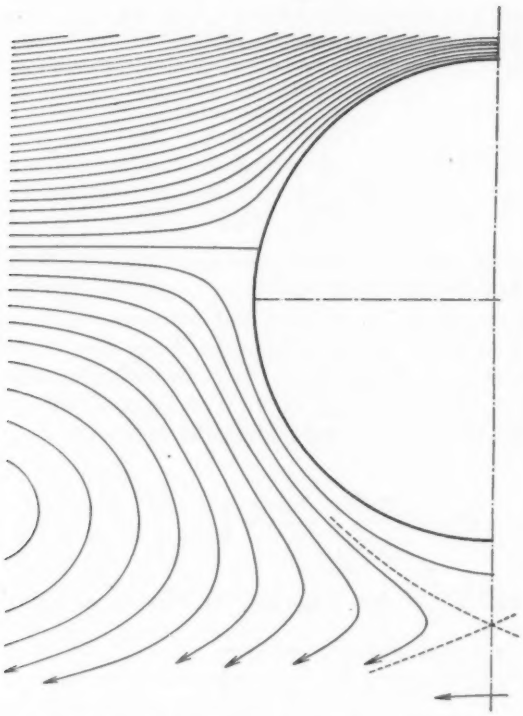

FIG. 5. Circular Cylinder in a Shear Flow with $K=2$.

which is a constant, verifying the boundary condition of the problem. The stagnation point on the cylinder can be calculated by means of Eq. (18). The condition is $(\partial\psi/\partial r)_{r=c/2} = 0$. This is satisfied at the point where

$$\sin\theta = \frac{1}{K}\left[\sqrt{1 + \frac{K^2}{4}} \pm 1\right]. \quad (20)$$

Therefore, if $-8/3 < K < 8/3$, there will be two stagnation points on the cyl-

inder located on that half of the surface where the velocity is higher. If K has a value beyond this range, there will be four stagnation points on the surface.

Fig. 5 shows the stream lines for the case of $K=2$. It is seen that there is an additional stagnation point in the flow field on the negative y-axis.

4. Force and moment. In this section, the force and the moment acting on a body whose disturbance stream function ψ_1 is given by Eq. (16) will be calculated.

The pressure p in the fluid is related to the velocity components u, v and the density ρ by Euler's equations

$$\begin{aligned} \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} &= -\frac{\partial p}{\partial x} \\ \rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} &= -\frac{\partial p}{\partial y}. \end{aligned} \tag{21}$$

By means of Eqs. (4), (7) and (16), the quantities on the left hand side of Eq. (21) can be calculated and then p can be obtained by integration. However in the final calculation of force and moment, the pressure will be integrated along a contour far away from the body; only terms up to $1/r^2$ in p need to be considered. Furthermore, the calculation can be simplified to a certain extent by differentiating the first of Eq. (21) with respect to x and the second with respect to y and adding the resultant. Then by using Eqs. (3) and (5),

$$\frac{1}{\rho}\nabla^2 p = 2\left[\frac{\partial^2\psi_1}{\partial x^2}\frac{\partial^2\psi_1}{\partial y^2} - \left(\frac{\partial^2\psi_1}{\partial x \partial y}\right)^2 + \frac{U_0 K}{c}\frac{\partial^2\psi_1}{\partial x^2}\right]. \tag{22}$$

By substituting Eq. (16) into Eq. (22), the differential equation for p is then

$$\begin{aligned} \frac{1}{\rho}\nabla^2 p &= \frac{1}{\rho}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2}\right] \\ &= 2U_0^2\left[-\frac{1}{r^4}(a_0^2 + b_0^2) + \frac{K}{c}\left\{\frac{1}{r^2}(-a_0\cos 2\theta + b_0 \sin 2\theta)\right.\right. \\ &\quad + \frac{1}{r^3}(2a_1 \cos 3\theta + 2b_1 \sin 3\theta) \\ &\quad \left.\left. + \frac{1}{r^4}(6a_2 \cos 4\theta + 6b_2 \sin 4\theta) + \cdots\right\}\right]. \end{aligned} \tag{23}$$

The appropriate solution for this non-homogeneous equation is evidently

$$\frac{1}{\rho}p = 2U_0^2\Big[-\frac{1}{4r^2}(a_0^2+b_0^2)+\frac{K}{c}\Big\{-\frac{1}{4}(-a_0\cos 2\theta+b_0\sin 2\theta)$$
$$-\frac{1}{8r}(2a_1\cos 3\theta+2b_1\sin 3\theta)$$
$$-\frac{1}{12r^2}(6a_2\cos 4\theta+6b_2\sin 4\theta)\Big\}\Big]$$
$$+A+A_0\log r+B_0\theta+\frac{1}{r}(A_1\cos\theta+B_1\sin\theta)$$
$$+\frac{1}{r^2}(A_2\cos 2\theta+B_2\sin 2\theta)+\cdots \tag{24}$$

where the A's and B's are undetermined constants. Either the first or the second equation (21) can be used to determine $A_0, A_1, \cdots$ and $B_0, B_1, B_2, \cdots$. The final result for the pressure p can be written as

$$\frac{p}{\rho} = A+U_0^2\Big[\frac{K}{c}a_0\log r+\frac{K}{c}b_0\theta-\frac{K}{2c}(-a_0\cos 2\theta+b_0\sin 2\theta)$$
$$-\frac{K}{2cr}(a_1\cos 3\theta+b_1\sin 3\theta)$$
$$+\frac{1}{r}\Big\{\Big(\frac{3}{2}\frac{Ka_1}{c}-b_0\Big)\cos\theta+\Big(\frac{3}{2}\frac{Kb_1}{c}-a_0\Big)\sin\theta\Big\}$$
$$+\frac{1}{r^2}\Big\{-\frac{a_0^2+b_0^2}{2}-\frac{K}{c}(a_2\cos 4\theta+b_2\sin 4\theta)$$
$$+\Big(2\frac{Ka_2}{c}-b_1\Big)\cos 2\theta+2\Big(\frac{Kb_2}{c}+a_1\Big)\sin 2\theta\Big\}+\cdots\Big]. \tag{25}$$

Now by considering the pressure force and momentum of the fluid, the following relations can be obtained between the forces X, Y acting on the body and their moment M_0 about the origin[2] (Fig. 1).

$$X = -\int_C p\,dy-\int_C \rho u(u\,dy-v\,dx)$$
$$Y = \int_C p\,dx-\int_C \rho v(u\,dy-v\,dx) \tag{26}$$
$$M_0 = \int_C p(x\,dx+y\,dy)-\int_C \rho(vx-uy)(u\,dy-v\,dx).$$

[2] See for example Glauert, H., *Aerofoil and airscrew theory*, Cambridge University Press, London, 1930, p. 80.

The integrals are taken along any closed curve C enclosing the body. If a circle with radius R is taken as the contour, then in the integrals of Eq. (26)

$$\begin{aligned} x &= R\cos\theta \\ y &= R\sin\theta. \end{aligned} \tag{27}$$

Therefore, by using Eq. (25),

$$-\int_C p\,dy = -R\int_0^{2\pi} p\cos\theta\,d\theta = \pi\rho U_0^2\left[b_0 - \frac{3}{2}\frac{K}{c}a_1\right]. \tag{28}$$

Furthermore,

$$\begin{aligned} u\,dy - v\,dx = U_0\Big[&\frac{K}{2c}R^2\sin 2\theta + R\cos\theta + b_0 + \frac{1}{R}(-a_1\sin\theta + b_1\cos\theta) \\ &+ \frac{1}{R^2}(-2a_2\sin 2\theta + 2b_2\cos 2\theta) + \cdots\Big]d\theta \end{aligned} \tag{29}$$

and

$$\begin{aligned} u = U_0\Big[&\frac{K}{c}R\sin\theta + 1 + \frac{1}{R}(a_0\sin\theta + b_0\cos\theta) \\ &+ \frac{1}{R^2}(-a_1\sin 2\theta + b_1\cos 2\theta) + \cdots\Big]. \end{aligned} \tag{30}$$

By combining Eqs. (29) and (30), the second term in the equation for the horizontal force X can be calculated as

$$-\int_C \rho u(u\,dy - v\,dx) = -\pi\rho U_0^2\left[3b_0 - \frac{3K}{2c}a_1\right]. \tag{31}$$

Finally the horizontal force is expressed as

$$X = -2\pi\rho U_0^2 b_0. \tag{32}$$

Similarly, the vertical force Y or lift and the moment M_0 about the origin are obtained in the following forms:

$$Y = 2\pi\rho U_0^2\left[a_0 + \frac{K}{c}(Rb_0 - b_1)\right] \tag{33}$$

$$M_0 = 2\pi\rho U_0^2\left[a_0 b_0 - a_1 + \frac{K}{c}(\tfrac{1}{2}R^2 b_0 - b_2)\right]. \tag{34}$$

Eqs. (32), (33) and (34) show that the force and moment on the body can be calculated in terms of the strength of the vortex, the source, the doublets

and the quadruplets in the disturbance stream function ψ_1. These equations can be regarded as the generalization of the well-known Blasius formulas. They reduce to the latter formulas if $K=0$. However, it should be noticed that if there is a source, i.e., $b_0 \neq 0$, both the lift Y and the moment M_0 grow to infinite magnitude with $R \to \infty$. This confirms the surmise stated previously. If the boundary of the body is closed, b_0 must vanish, and there can be no horizontal force or drag and the lift and moment will remain finite. v. Sanden[1] obtained a small drag force for his wedge shaped body. Evidently this is due to the unavoidable inaccuracies in his numerical method.

5. Symmetrical Joukowsky airfoils. For the flow over a symmetrical Joukowsky airfoil, it is difficult to determine the disturbance stream function ψ_1 directly. But, as seen from Eq. (9), ψ_1 satisfies Laplace's equation, and therefore conformal transformation can be used. It should be noticed, however, that ψ_0 does not satisfy Laplace's equation; and therefore it does not allow the use of this transformation in the ordinary sense.

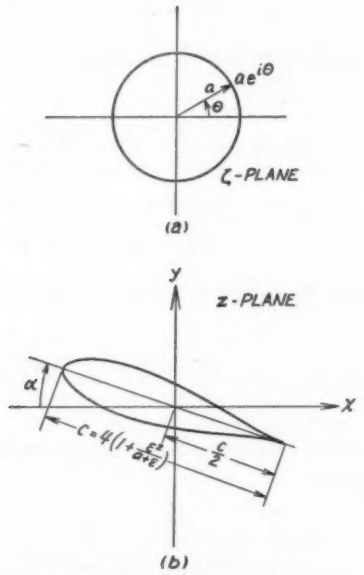

FIG. 6. Joukowsky Transformation of a Circle into an Airfoil.

Consider a circle of radius a in the ζ plane (Fig. 6a). The transformation

$$z = e^{i\alpha}\left(\zeta - \epsilon + \frac{1}{\zeta - \epsilon} + \frac{2\epsilon^2}{a + \epsilon}\right) \tag{35}$$

will transform the circle into a symmetrical Joukowsky airfoil in the z plane

(Fig. 6b) inclined at an angle α against the x-axis, ϵ is a positive quantity which increases with the thickness of the airfoil. ϵ is zero for a flat plate and is infinite for a circle. Furthermore

$$a = 1 + \epsilon. \tag{36}$$

The origin of the z-plane lies at the center of the airfoil (Fig. 6b). If we represent the circle in the ζ-plane by

$$\zeta = ae^{i\theta}, \tag{37}$$

the trailing edge of the airfoil corresponds to the point where $\theta = 0$. The leading edge of the airfoil corresponds to the point where $\theta = \pi$. Therefore, the chord is

$$c = 4\left(1 + \frac{\epsilon^2}{a + \epsilon}\right). \tag{38}$$

With

$$\epsilon/a = \lambda$$

the values of x and y corresponding to an arbitrary value of θ are given by Eq. (35) as

$$\begin{aligned} x &= a\left\{\cos(\theta - \alpha) - \lambda\cos\alpha + \frac{\cos(\theta + \alpha) - \lambda\cos\alpha}{a^2(1 - 2\lambda\cos\theta + \lambda^2)} + \frac{2\lambda^2\cos\alpha}{1 + \lambda}\right\} \\ y &= a\left\{\sin(\theta - \alpha) + \lambda\sin\alpha - \frac{\sin(\theta + \alpha) - \lambda\sin\theta}{a^2(1 - 2\lambda\cos\theta + \lambda^2)} - \frac{2\lambda^2\sin\alpha}{1 + \lambda}\right\}. \end{aligned} \tag{39}$$

After Eq. (7), the velocity due to ψ_0 at a point in the z-plane corresponding to $\zeta = ae^{i\theta}$ is

$$U = U_0\left[1 + \frac{Ka}{c}\left\{\sin(\theta - \alpha) - \frac{\sin(\theta + \alpha) - \lambda\sin\alpha}{a^2(1 - 2\lambda\cos\theta + \lambda^2)} + \frac{\lambda\sin\alpha}{a(1 + \lambda)}\right\}\right] \tag{40}$$

This velocity is horizontal and has a tangential component in the counterclockwise direction equal to

$$-U\left[\frac{dx}{\sqrt{(dx)^2 + (dy)^2}}\right]_{\zeta = ae^{i\theta}} \tag{41}$$

and a normal component, directed along the outward normal, equal to

$$U\left[\frac{dy}{\sqrt{(dx)^2 + (dy)^2}}\right]_{\zeta = ae^{i\theta}}. \tag{42}$$

On the other hand if ψ_1 is the disturbance stream function, the velocity component normal to the circle $\zeta = ae^{i\theta}$ is

$$\left(\frac{1}{r}\frac{\partial\psi_1}{\partial\theta}\right)_{r=a}. \tag{43}$$

Therefore, the corresponding velocity component in the z-plane, normal to the surface of the airfoil, is

$$\left(\frac{1}{r}\frac{\partial\psi_1}{\partial\theta}\right)_{r=a}\left|\frac{d\zeta}{dz}\right|_{\zeta=ae^{i\theta}}. \tag{44}$$

Then the boundary condition at the airfoil surface requires that this normal component of the disturbance velocity be equal to the negative of the normal velocity component due to ψ_0. By means of Eqs. (42) and (44), this boundary condition is expressed by

$$\left(\frac{1}{r}\frac{\partial\psi_1}{\partial\theta}\right)_{r=a}\left|\frac{d\zeta}{dz}\right|_{\zeta=ae^{i\theta}} = -U\left[\frac{dy}{\sqrt{(dx)^2+(dy)^2}}\right]_{\zeta=ae^{i\theta}}. \tag{45}$$

But

$$\left[\sqrt{(dx)^2+(dy)^2}\right]_{\zeta=ae^{i\theta}} = \frac{ad\theta}{\left|\frac{d\zeta}{dz}\right|_{\zeta=ae^{i\theta}}}. \tag{46}$$

With the aid of Eqs. (39), (40), and (42), Eq. (45) can be re-written as follows

$$\begin{aligned}\frac{1}{U_0}&\left(\frac{1}{r}\frac{\partial\psi_1}{\partial\theta}\right)_{r=a}\\ &= -\left[1+\frac{Ka}{c}\left\{\sin(\theta-\alpha)-\frac{\sin(\theta+\alpha)-\lambda\cos\alpha}{a^2(1-2\lambda\cos\theta+\lambda^2)}+\frac{\lambda\sin\alpha}{a(1+\lambda)}\right\}\right]\\ &\quad\cdot\left[\cos(\theta-\alpha)-\frac{\cos(\theta+\alpha)+\lambda\cos(\theta-\alpha)+2\lambda\cos\alpha}{a^2(1-2\lambda\cos\theta+\lambda^2)^2}\right].\end{aligned} \tag{47}$$

This equation alone would not determine the function ψ_1 completely but for the additional so-called Kutta-Joukowsky condition, which fixes the strength of the circulation over the airfoil.

6. Strength of circulation. The Kutta-Joukowsky condition states that the velocity at the trailing edge of the airfoil must be finite. The velocity at the trailing edge of the airfoil consists of two parts: one part is that due to ψ_0 and the other that due to ψ_1. Only tangential components need be considered because the normal components cancel each other as required by the boundary condition. The part due to ψ_1 in the counterclockwise direction is

$$-\frac{\partial\psi_1}{\partial r}\left|\frac{d\zeta}{dz}\right|_{\zeta=a}. \tag{48}$$

Therefore, by means of Eq. (41), the resultant velocity at the trailing edge is

$$\left[-\frac{\partial \psi_1}{\partial r} - U \frac{dx}{\sqrt{(dx)^2 + (dy)^2}} \frac{1}{\left|\frac{d\zeta}{dz}\right|_{\zeta=a}} \right] \left|\frac{d\zeta}{dz}\right|_{\zeta=a}; \tag{49}$$

but $\zeta = a$ is the singular point of the transformation from ζ to z as can be easily verified. In other words, $|d\zeta/dz|$ tends to infinity at $\zeta = a$. Therefore, the resultant velocity at the trailing edge can only be finite if the quantity within the square bracket of Eq. (49) vanishes. By means of Eqs. (39), (40), (46), this condition can be written as

$$-\left(\frac{\partial \psi_1}{\partial r}\right)_{\substack{r=a \\ \theta=0}} - U_0 \left[1 + \frac{Ka}{c} \left\{ \sin(\theta-\alpha) - \frac{\sin(\theta+\alpha) - \lambda \cos\alpha}{a^2(1-2\lambda\cos\theta+\lambda^2)} + \frac{\lambda \sin\alpha}{a(1+\lambda)} \right\} \right] \cdot \left[\sin(\theta-\alpha) + \frac{\sin(\theta+\alpha) - \lambda^2 \sin(\theta-\alpha) - 2\lambda \sin\alpha}{a^2(1-2\lambda\cos\theta+\lambda^2)} \right]_{\theta=0} = 0. \tag{50}$$

The appropriate general solution ψ_1 in the ζ-plane is

$$\psi_1 = U_0 \left[\alpha_0 \log r + \beta_0 \theta + \sum_{n=1}^{\infty} (\alpha_n \cos n\theta + \beta_n \sin n\theta) \frac{1}{r^n} \right]. \tag{51}$$

Therefore,

$$\begin{aligned} \frac{1}{U_0}\left(\frac{1}{r}\frac{\partial \psi_1}{\partial \theta}\right)_{r=a} &= \frac{\beta_0}{a} + \sum_{n=1}^{\infty} (-n\alpha_n \sin n\theta + n\beta_n \cos n\theta) \frac{1}{a^{n+1}}, \\ -\frac{1}{U_0}\left(\frac{\partial \psi_1}{\partial r}\right)_{r=a} &= -\frac{\alpha_0}{a} + \sum_{n=1}^{\infty} (n\alpha_n \cos n\theta + n\beta_n \sin n\theta) \frac{1}{a^{n+1}}. \end{aligned} \tag{52}$$

By expanding the right hand side of Eq. (47) into a trigonometric series, all the coefficients $\alpha_1, \alpha_2, \cdots$ and $\beta_0, \beta_1, \cdots$ in ψ_1 can be determined by the first of Eqs. (52). Then the second equation of Eqs. (52) together with Eq. (50) will determine the value of α_0. Actually, calculations are easier if each term on the right of Eq. (47) is taken separately. For example, the term $-U_0 \cos(\theta-\alpha)$ can be expanded into $-U_0 \cos\alpha \cos\theta - U_0 \sin\alpha \sin\theta$. Then according to Eq. (52), the contribution to α_1 is $a^2 \sin\alpha$; the contribution to β_1 is $-a^2 \cos\alpha$. Finally the contribution to $-(\partial\psi_1/\partial r)_{r=a}$ is $U_0(\sin\alpha\cos\theta - \cos\alpha\sin\theta) = -U_0 \sin(\theta-\alpha)$. The other more complicated terms can be treated in a similar manner with the aid of the following expansions and their derivatives with respect to θ and λ:

$$\begin{aligned} \frac{\sin\theta}{1-2\lambda\cos\theta+\lambda^2} &= \sum_{n=1}^{\infty} \lambda^{n-1} \sin n\theta \\ \frac{\cos\theta-\lambda}{1-2\lambda\cos\theta+\lambda^2} &= \sum_{n=1}^{\infty} \lambda^{n-1} \cos n\theta \\ \frac{1}{1-2\lambda\cos\theta+\lambda^2} &= \frac{1}{1-\lambda^2}\left(1 + 2\sum_{n=1}^{\infty} \lambda^n \cos n\theta\right). \end{aligned}$$

The value of $-(\partial\psi_1/\partial r)_{r=a}$ thus found is

$$-\frac{1}{U_0}\left(\frac{\partial\psi_1}{\partial r}\right)_{r=a}=-\frac{\alpha_0}{a}-\left[1+\frac{K\lambda}{c(1+\lambda)}\right]$$
$$\cdot\left[\sin(\theta-\alpha)-\frac{\sin(\theta+\alpha)-\lambda^2\sin(\theta-\alpha)-2\lambda\sin\alpha}{a^2(1-2\lambda\cos\theta+\lambda^2)}\right]$$
$$+\frac{Ka}{c}\left[\tfrac{1}{2}\cos 2(\theta-\alpha)+\left\{\sin(\theta-\alpha)-\frac{\sin(\theta+\alpha)-\lambda\cos\alpha}{a^2(1-2\lambda\cos\theta+\lambda^2)}\right\}\cdot\right.$$
$$\cdot\frac{\sin(\theta+\alpha)-\lambda^2\sin(\theta-\alpha)-2\lambda\sin\alpha}{a^2(1-2\lambda\cos\theta+\lambda^2)^2}$$
$$\left.-\frac{\cos(\theta-\alpha)\left[\cos(\theta+\alpha)-\lambda\cos\alpha\right]}{a^2(1-2\lambda\cos\theta+\lambda^2)}+\frac{1}{2a^4(1-\lambda^2)(1-2\lambda\cos\theta+\lambda^2)}\right]. \quad (53)$$

The strength of circulation α_0 necessary to satisfy the Kutta-Joukowsky condition can then be calculated by means of Eq. (50). The result is

$$\alpha_0=aU_0\left[2\sin\alpha+\frac{Ka}{c}\left\{\frac{2\lambda\sin^2\alpha}{a(1+\lambda)}-\tfrac{1}{2}+\cos 2\alpha-\frac{1}{a}+\frac{1}{2a(1+\lambda)}\right\}\right] \quad (54)$$

The strength of the source, β_0, is zero, as would be expected from the fact that the airfoil is represented by a closed contour.

7. Strengths of doublets and quadruplets. By collecting the contributions of the different terms of $[(1/r)\,\partial\psi_1/\partial\theta]_{r=a}$ in Eq. (47) to α_1, α_2, β_1 and β_2, the following values of the strength of doublets and quadruplets in the ζ-plane are obtained

$$\alpha_1=a^2U_0\left[\left(1+\frac{1}{a^2}\right)\sin\alpha\right.$$
$$\left.+\frac{Ka}{c}\left\{\frac{\lambda}{2a^2}\cos 2\alpha-\frac{\lambda}{2a^4(1-\lambda^2)}+\left(1+\frac{1}{a^2}\right)\frac{\lambda\sin^2\alpha}{a(1+\lambda)}\right\}\right]$$
$$\beta_1=a^2U_0\left[\left(-1+\frac{1}{a^2}\right)\cos\alpha+\frac{Ka}{c}\sin 2\alpha\left\{-\frac{\lambda}{2a^2}+\left(-1+\frac{1}{a^2}\right)\frac{\lambda}{2a(1+\lambda)}\right\}\right]$$
$$\alpha_2=a^3U_0\left[\frac{\lambda}{a^2}\sin\alpha+\frac{Ka}{c}\cos 2\alpha\left\{\frac{1}{4}-\frac{1}{2a^2}+\frac{\lambda^2}{2a^2}+\frac{1-3\lambda^2}{4a^4(1-\lambda^2)}\right\}\right]$$
$$\beta_2=a^3U_0\left[\frac{\lambda}{a^2}\cos\alpha+\frac{Ka}{c}\sin 2\alpha\left\{\frac{1}{4}-\frac{\lambda^2}{2a^2}-\frac{1-3\lambda^2}{4a^4(1-\lambda^2)}\right\}\right]. \quad (55)$$

However, to calculate the lift and moment over the airfoil, it is not the strength of doublets and quadruplets in the ζ-plane that is needed but the strength of those in the z-plane, where the airfoil is located. From the known

values of $\alpha_1, \cdots \beta_2$ given by Eq. (35) and the transformation specified by Eq. (55), the desired quantities can be easily calculated in the following manner.

By introducing the conjugate function ϕ_1 of the stream function one can write

$$\phi_1 + i\psi_1 = i\alpha_0 \log \zeta + \frac{-\beta_1 + i\alpha_1}{\zeta} + \frac{-\beta_2 + i\alpha_2}{\zeta^2} + \cdots . \tag{56}$$

With $\zeta = re^{i\theta}$, the real part of Eq. (56) is Eq. (51) since β_0 is found to be zero. Now Eq. (35) gives

$$\zeta = ze^{i\alpha}\left[1 + \frac{\lambda}{(1+\lambda)ze^{i\alpha}} - \frac{1}{z^2e^{2i\alpha}} + \cdots\right] \tag{57}$$

for sufficiently large values of z. Then $\phi_1 + i\psi_1$ can be expanded into a series in z by substituting Eq. (57) into Eq. (56). The result is

$$\phi_1 + i\psi_1 = i\alpha_0 \log z + \left[-\beta_1 + i\left(\alpha_1 + \frac{\lambda\alpha_0}{1+\lambda}\right)\right] z^{-1}e^{-i\alpha}$$
$$+ \left[-\beta_2 + \frac{\lambda\beta_1}{1+\lambda} + i\left(\alpha_2 - \frac{\lambda\alpha_1}{1+\lambda} - \alpha_0 - \frac{\lambda^2\alpha_0}{2(1+\lambda)^2}\right)\right] z^{-2}e^{-2i\alpha} + \cdots . \tag{58}$$

If the strength of the circulation, the doublets, and the quadruplets in the z-plane are denoted by a_0, a_1, b_1, a_2 and b_2, respectively, the following relations are obtained by comparing Eq. (58) with Eq. (56):

$$\begin{aligned}
a_0 &= \alpha_0, \qquad b_0 = 0 \\
a_1 &= \beta_1 \sin\alpha + \left(\alpha_1 + \frac{\lambda\alpha_0}{1+\lambda}\right)\cos\alpha \\
b_1 &= \beta_1 \cos\alpha - \left(\alpha_1 + \frac{\lambda\alpha_0}{1+\lambda}\right)\sin\alpha \\
a_2 &= \left(\beta_2 - \frac{\lambda\beta_1}{1+\lambda}\right)\sin 2\alpha + \left(\alpha_2 - \frac{\lambda\alpha_1}{1+\lambda} - \alpha_0 - \frac{\lambda^2\alpha_0}{2(1+\lambda)^2}\right)\cos 2\alpha \\
b_2 &= \left(\beta_2 - \frac{\lambda\beta_1}{1+\lambda}\right)\cos 2\alpha - \left(\alpha_2 - \frac{\lambda\alpha_1}{1+\lambda} - \alpha_0 - \frac{\lambda^2\alpha_0}{2(1+\lambda)^2}\right)\sin 2\alpha.
\end{aligned} \tag{59}$$

Eqs. (54), (55), and (59) give all the necessary data to calculate the forces and moment acting on the airfoil.

8. Lift and moment coefficients. By means of Eqs. (32), (33), and (34), the drag, lift, and moment about the origin can be calculated with the values of a_0, a_1, a_2, b_0, b_1, b_2 given by Eq. (59). It is seen that the drag is also zero in

the case of shear flow. The result of the computation can be conveniently represented by defining the functions l_0, l_1, l_2, l_3, l_4 and m_0, m_1, m_2, m_3, m_4 in connection with the lift coefficient C_L and moment coefficient C_{M_0} in the following manner:

$$C_L=\frac{Y}{\frac{1}{2}\rho U_0^2 c}=2\pi\left[l_0 \sin\alpha+K\{l_1+l_2\cos 2\alpha\}+K^2\{l_3\sin\alpha+l_4\sin 3\alpha\}\right]$$

$$C_{M_0}=\frac{-M_0}{\frac{1}{2}\rho U_0^2 c^2}=\frac{\pi}{2}\left[\frac{m_0}{2}\sin 2\alpha+K\{m_1\cos\alpha+m_2\cos 3\alpha\}\right. \tag{60}$$

$$\left.+K^2\{m_3\sin\alpha+m_4\sin 4\alpha\}\right].$$

The negative sign in the moment coefficient is introduced in accordance with the usual convention of taking the stalling moment as positive. The functions l_0, l_1, l_2, l_3, l_4 and m_0, m_1, m_2, m_3, m_4 are given by the following equations:

$$\left.\begin{aligned} l_0&=\frac{1}{h}, & h&=\frac{1}{a}+\frac{\lambda^2}{1+\lambda}\\ l_1&=\frac{\lambda(\lambda+2/a)}{8h^2(1+\lambda)}, & l_2&=\frac{\lambda^2(2+1/a)}{8h^2(1+\lambda)}\\ l_3&=-\frac{\lambda}{64h^3a(1+\lambda)}\left[\frac{1}{a^2(1+\lambda)}-2\lambda+\lambda^2\right]\\ l_4&=\frac{\lambda}{64a^2h^2}\left[1-\frac{\lambda}{(1+\lambda)^2}\right] \end{aligned}\right\} \tag{61}$$

$$\left.\begin{aligned} m_0&=\frac{1}{ah^2}\left[\frac{1}{a}+\frac{\lambda}{1+\lambda}\right]\\ m_1&=\frac{1}{8ah^3}\left[\frac{1}{a}+\frac{\lambda}{2(1+\lambda)}\left(1+\lambda+\frac{\lambda}{a(1+\lambda)}\right)\right]\\ m_2&=-\frac{1}{8ah^3}\left[\frac{1}{a}(1-\tfrac{3}{2}\lambda)+\frac{\lambda}{2(1+\lambda)}\left(\frac{2}{a^2}-\lambda^2\frac{2+1/a}{1+\lambda}\right)\right]\\ m_3&=-\frac{1}{64ah^4}\left[\frac{1}{1+\lambda}\left(\frac{1}{a^2}+\frac{\lambda^2}{2a^2(1+\lambda)^2}\right)-\frac{\lambda^2}{2a(1+\lambda)^2}\right]\\ m_4&=\frac{1}{64ah^4}\left[1+\lambda-\frac{\lambda}{a^2(1+\lambda)}\left(1+\frac{\lambda}{a(1+\lambda)}\right)+\frac{\lambda^2(1+\lambda^2)}{2a(1+\lambda)(1-\lambda^2)}\right.\\ &\quad\left.-\frac{1}{4a^2}\left(\frac{1}{a}+\frac{1}{1+\lambda}\right)+\frac{9\lambda^2}{4a^2(1+\lambda)}\right]. \end{aligned}\right\} \tag{62}$$

Table I gives the numerical values of these functions together with the thickness ratio δ of the airfoil. It is seen that with increasing thickness the

TABLE I

ϵ	δ	l_0	l_1	l_2	l_3	l_4	m_0	m_1	m_2	m_3	m_4
0	0	1.0000	0	0	0	0	1.0000	0.1250	−0.1250	−0.0156	0.0078
0.05	0.0618	1.0476	0.0172	0.0009	−0.0006	0.0007	1.0430	0.1334	−0.1267	−0.0155	0.0102
0.10	0.1179	1.0909	0.0237	0.0033	−0.0009	0.0014	1.0737	0.1413	−0.1259	−0.0152	0.0128
0.15	0.1687	1.1304	0.0345	0.0069	−0.0010	0.0020	1.0945	0.1477	−0.1231	−0.0148	0.0157
0.20	0.2150	1.1667	0.0446	0.0115	−0.0009	0.0025	1.1073	0.1530	−0.1190	−0.0143	0.0188
0.30	0.2958	1.2308	0.0628	0.0227	−0.0003	0.0034	1.1148	0.1610	−0.1080	−0.0131	0.0250
0.40	0.3636	1.2857	0.0787	0.0356	+0.0005	0.0040	1.1058	0.1660	−0.0953	−0.0119	0.0310
0.50	0.4210	1.3333	0.0926	0.0494	+0.0014	0.0045	1.0864	0.1687	−0.0823	−0.0106	0.0365
0.60	0.4701	1.3750	0.1047	0.0635	+0.0023	0.0048	1.0608	0.1697	−0.0697	−0.0095	0.0416
0.80	0.5489	1.4444	0.1248	0.0911	+0.0039	0.0051	1.0006	0.1683	−0.0474	−0.0075	0.0499
1.00	0.6089	1.5000	0.1406	0.1172	+0.0051	0.0051	0.9375	0.1641	−0.0293	−0.0059	0.0560
∞	1.0000	2.0000	0.2500	0.5000	0	0	0	0	0	0	0

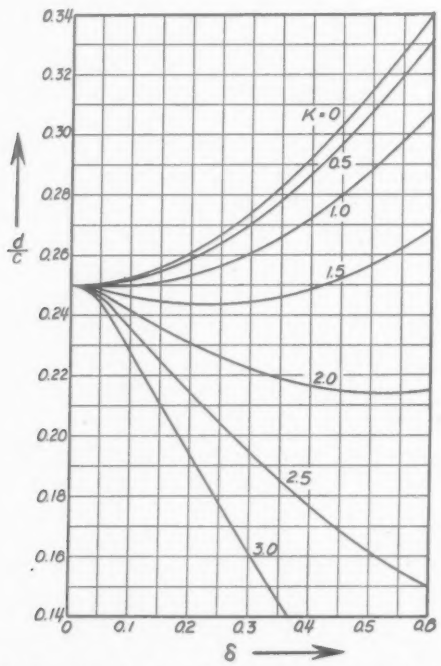

FIG. 7. Ratio of the distance d from leading edge to aerodynamic center (positive when it is behind the leading edge) to the chord ϵ as a function of thickness ratio δ and non-dimensional velocity gradient K.

effect of the velocity gradient K becomes larger and larger. For example, if $\delta = 11.79\%$ the effect of K on C_L at small angles of attack is approximately given by the term $2\pi K(l_1+l_2) = 2\pi K \times 0.02695$. In other words, it is equivalent to a shift in the angle of zero lift by 1.54 K degrees. For an airfoil of 21.50% thickness, this value is 3.21 K degrees.

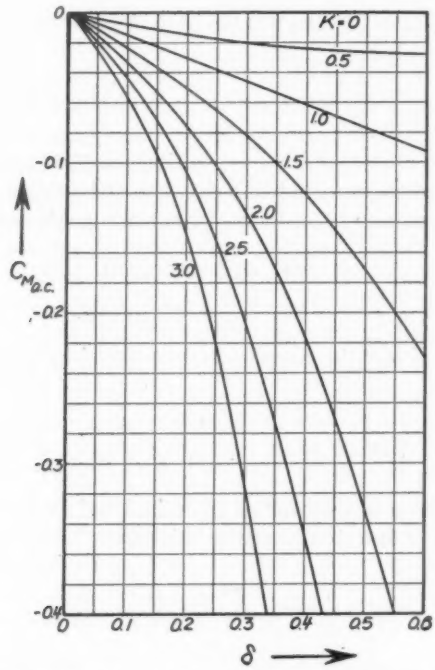

FIG. 8. Moment coefficient $C_{M_{a.c}}$ about the aerodynamic center (positive for stalling moment) as a function of thickness ratio and non-dimensional velocity gradient K.

9. Aerodynamic center. To demonstrate the effect of the velocity gradient on the moment more clearly, the aerodynamic center for small angles of attack will be calculated presently. For small values of α the expressions for C_L and C_{M_0} of Eq. (60) can be simplified to

$$\begin{aligned} C_L &= 2\pi[\{l_0 + K^2(l_3 + 3l_4)\}\alpha + K(l_1 + l_2)] \\ C_{M_0} &= \frac{\pi}{2}[\{m_0 + K^2(2m_3 + 4m_4)\}\alpha + K(m_1 + m_2)]. \end{aligned} \tag{63}$$

The moment coefficient corresponds to the stalling moment about the center of the airfoil. If the moment is referred to a point on the chord at a

distance d back of the leading edge, the corresponding moment coefficient C_M will be

$$\begin{aligned} C_M &= C_{M_0} - C_L\left(\frac{1}{2} - \frac{d}{c}\right) \\ &= \frac{\pi}{2}\left[\{m_0 + K^2(2m_3 + 4m_4)\}\alpha + K(m_1 + m_2)\right] \\ &\quad - \left(\frac{1}{2} - \frac{d}{c}\right)2\pi\left[\{l_0 + K^2(l_3 + 3l_4)\}\alpha + K(l_1 + l_2)\right]. \end{aligned} \tag{64}$$

At the aerodynamic center, the corresponding moment coefficient should be independent of the angle of attack α. From Eq. (64), this condition gives the distance d of aerodynamic center back of the leading edge as

$$\frac{d}{c} = \frac{1}{2} - \frac{1}{4}\,\frac{m_0 + K^2(2m_3 + 4m_4)}{l_0 + K^2(l_3 + 3l_4)}\,. \tag{65}$$

Then the moment about the aerodynamic center is given by the following coefficient:

$$C_{M_{a.c.}} = \frac{\pi K}{2}\left[(m_1 + m_2) - \frac{m_0 + K^2(2m_3 + 4m_4)}{l_0 + K^2(l_3 + 3l_4)}\,(l_1 + l_2)\right]. \tag{66}$$

The numerical values of d/c and $C_{M_{a.c.}}$ calculated from Eqs. (65) and (66) are plotted in Figs. 7 and 8 against the thickness ratio δ for different values of K.

ON THE DISCONTINUOUS FLOW AROUND AN AIRFOIL WITH FLAP*

BY

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Summary. A method is given for calculating the effect of the wake on the pressure distribution produced by the steady, two dimensional flow around an airfoil with infinite span and broken line profile. To determine the size of the wake there is proposed the hypothesis that the flow is such as to create a minimum disturbance at distant points.

1. Introduction. The general Navier-Stokes equations for the steady, viscous flow of an incompressible fluid around an immersed body have not been solved even for such simple cases as the sphere or cylinder. However, it is well known that certain aspects of the flow are adequately represented by neglecting viscosity entirely. Thus the perfect fluid theory of Kutta and Joukowski for calculating the reaction of the air on an airfoil with infinite span, engaged in a uniform rectilinear translation through a bulk of air at rest, gives satisfactory results so far as the lift is concerned. This theory fails to account for all the details of the flow, however. In particular, it neglects the wake of deadwater which exists at the trailing edge. The effect of this wake is to make the lift slightly smaller than the calculated value and to reduce the negative pressures along the rear portion of the suction side. Since the effect is greatest near the trailing edge, the discrepancies between calculations and observation will be most pronounced for the total hinge moment of the pressures on a trailing edge flap.

The boundary layer theory has been used to calculate the size of the wake, i.e. the point where the wake detaches itself from the body. Basing his calculations on the observed pressure distribution, Hiemenz in 1911 obtained from the boundary layer theory an angle of 82° from the forward stagnation point for the point of detachment in flow around an infinite circular cylinder, a value in good agreement with his experiments. However, his experiments were made at low Reynolds numbers; an airfoil would be stalled for such a flow. For the high Reynolds numbers corresponding to the normal, unstalled flight of an airfoil, the calculated point of detachment for flow around the cylinder is at least 20° smaller than the observed value, even though the calculations are based on experimental pressure distributions.

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A definite advance was made by C. Schmieden in 1932[1] when he showed that for high Reynolds numbers the perfect fluid theory is adequate to explain the wake behind a cylinder if one assumes a discontinuous flow. According to this view, the wake is obtained as a special case of the classical theory of jets. The theory of jets, due to Helmholtz and Levi-Civita, assumes a wake of stationary fluid at uniform pressure, bounded by two streamlines along which the speed is constant and different from zero. In spite of the fact that such a discontinuity in velocities would be impossible in a real fluid, Schmieden's calculated values of the pressures around the cylinder agree remarkably well with experiment. The agreement is perfect up to the pressure minimum; beyond that there are some deviations, which can be attributed to turbulence in the boundary layer and circulation of fluid in the wake. The point of detachment is approximately correct, but the observed point is somewhat masked by the turbulence; and the circulation of the fluid in the wake causes the constant pressure in the wake to be negative instead of zero as predicted by the theory.

Schmieden found, however, that the equations of motion for a perfect fluid plus the usual boundary condition that the velocity approach the stream velocity as the distance from the cylinder increases without bound are insufficient to determine the discontinuous flow uniquely. Various sizes and types of wakes are possible. There exist symmetric discontinuous flows with wakes detaching from the cylinder at every angle from about 55° to 180°. For angles between 55° and about 124° the wake extends to infinity and is bounded by streamlines which are first concave toward the wake and then, after passing through a point of inflection, are convex; the drag on the cylinder is positive; and the constant pressure in the wake is zero. For angles between 124° and 180° the wake is finite in extent, the drag is zero, and the pressure in the wake is positive.[2] For the critical angle of about 55° the wake boundaries are always convex (Helmholtz case). For the critical angle of about 124° the wake is infinite but has a width which tends asymptotically to zero, the boundaries of the wake are concave, the drag vanishes, and the pressure in the wake is zero; we shall refer to this as the *Schmieden case.*

Since a correctly set physical problem should have a unique answer even for a perfect fluid, an additional boundary condition is evidently required to determine the flow. Schmieden has chosen the auxiliary condition that the wake boundaries be concave and infinite in extent. A possible reason for selecting the condition of concavity is the intuitive feeling that in a viscous fluid any portion of a streamline with a point of inflection would be swept

[1] Schmieden, C., *Über die Eindeutigkeit der Lösungen in der Theorie der unstetigen Strömungen*, Ingenieur-Archiv 3, pages 356–370 (1932) and 5, 373–375 (1934).

[2] Schmieden considers only infinite wakes, but it is obvious that finite wakes can exist: see Kolscher, M., *Unstetige Strömungen mit endlichem Totwasser*, Luftfahrtforschung 17, 154–160 (1940). The present paper also restricts attention to infinite wakes.

away to infinity. However, for a perfect fluid there would be no such tendency; moreover, the condition of concavity is an "internal" condition, more in the nature of a law of motion than of a boundary condition. The required boundary condition must be "external" in character—a condition which can be imposed arbitrarily as a constraint on the system. Such an external condition could only be imposed by specifying something about the nature of the impinging stream. In particular we could require that this stream be as nearly uniform as possible, that is, the difference between the velocity and the undisturbed stream velocity shall approach zero as rapidly as possible as one recedes from the cylinder in the direction of the stream source. In the present paper we show that this assumption of "minimum disturbance at infinity" does in fact lead to the Schmieden case, at least for infinite wakes.

On the experimental side the justification for choosing the Schmieden case for an approximation to the flow of an actual fluid at high Reynolds numbers lies in the agreement previously mentioned between calculated and observed pressure distributions.[3] Of the various discontinuous flows the Schmieden case gives the best agreement for the pressure distribution as a whole, but more particularly for the forward part of the cylinder where turbulence is absent. As previously mentioned the observed pressure in the wake is negative instead of zero, but this is still the best agreement possible since the theory predicts a positive pressure for finite wakes. Quantative technical applications involving pressures behind the cylinder are subject to the limitations imposed by this discrepancy. Further refinements of the theory to eliminate this discrepancy by taking into account the turbulence in the wake have been suggested by Schmieden (Reference 1, page 364).

In 1940 Schmieden[4] applied his theory of discontinuous flow to the case of an airfoil with infinite span whose profile is an inclined straight line segment. It was found that the pressure distributions so obtained agree almost exactly with the pressure calculated from the Kutta-Joukowski theory except near the trailing edge, where the wake would be expected to exert an appreciable influence. It should be observed that the Kutta-Joukowski theory also uses an additional assumption beyond those involved in the perfect fluid equations, namely, the Joukowski hypothesis that the circulation is such that the velocity at the trailing edge is finite. Schmieden makes no direct assumption about the circulation, but replaces the Joukowski hypothesis by the above assumption concerning the nature of the wake. From Schmieden's point of view the Joukowski hypothesis is a special assumption concerning the nature of the wake in the limiting case when the wake is made to disappear.

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[3] Schmieden compares his results with the experimental values of Eisner, F., *Widerstandsmessungen an umströmten Zylindern*, Mitteilungen der Preussischen Versuchsanstalt für Wasserbau and Schiffbau, Heft 4, Berlin 1929.

[4] Schmieden, C. *Über Tragflügelströmungen mit Wirbelablösung*, Luftfahrtforschung **17**, 37–44 (1940).

An objection has been raised to Schmieden's theory of the airfoil with straight line profile in that the pressure is negatively infinite at the leading edge; it is one of the requirements of the classical theory of jets that separation of the fluid from the body occurs at such points. However, in an actual airfoil no separation occurs at the leading edge (unless the airfoil is stalled) because the leading edge is rounded off. The infinite velocity is not a necessary feature of the Schmieden theory, but only results from the sharp leading edge which was assumed to render the mathematics feasible. Singularities of exactly the same order are obtained from the Kutta-Joukowski theory when applied to sharp leading edges.

In the present paper we apply the Schmieden theory to the case of two dimensional flow around an airfoil with flap, the profile being a broken line. Some of the results are compared with the Kutta-Joukowski theory in Figs. 2 to 5.

2. The general results. Consider a plate whose cross section is a broken straight line (Fig. 1). The angle of attack of the main wing may be denoted

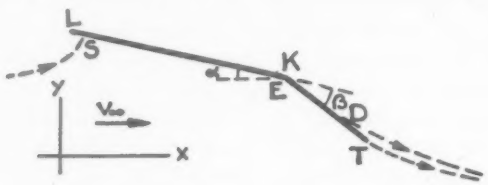

FIG. 1.

by α and the angle of attack of the flap by $\alpha+\beta$. Thus β is the angle of depression of the flap. We assume that α is positive and that β, if negative, is greater than $-\alpha$. The velocity of the undisturbed stream (velocity of flight) is denoted by v_∞.

There will be a critical streamline which divides at the stagnation point S, follows both sides of the airfoil to the points of detachment, D and T, and then to infinity as boundaries of the wake. At the points of detachment the boundaries of the wake leave the plate tangentially.

Evidently the ratio of the lengths (chords) $\overline{KT}$ to $\overline{LK}$ must not be too small, or else the point of detachment D will not fall on the flap. We restrict attention to cases where the flap is long enough for D to lie on the flap—an assumption that appears to be justified in all cases of technical interest.

The method of Levi-Civita* does not give the velocity distribution along the surface of the plate directly in terms of the position which the point under consideration has on the plate; but instead, both position and velocity

* For an exposition of the Levi-Civita theory in English see: Milne-Thomson, *Theoretical Hydrodynamics*, London, 1938, pp. 305–316.

are expressed parametrically in terms of an auxiliary variable ϕ, where ϕ is an angle ranging from 0 to π. The explicit expressions are as follows, where v is the speed of the fluid and s is the distance measured along the broken line from T. (For the proofs of these formulas see section 5):

$$v = v_\infty \frac{\sin \tfrac{1}{2}(\phi_L+\phi) \sin \tfrac{1}{2}\left|\phi_S-\phi\right|}{\sin \tfrac{1}{2}\left|\phi_L-\phi\right| \sin \tfrac{1}{2}(\phi_S+\phi)} \left[\frac{\sin \tfrac{1}{2}(\phi_K+\phi) \sin \tfrac{1}{2}\left|\phi_E-\phi\right|}{\sin \tfrac{1}{2}\left|\phi_K-\phi\right| \sin \tfrac{1}{2}(\phi_E+\phi)}\right]^{\beta/\pi} \tag{1}$$

$$s = 4c^2 \int_0^\phi \frac{\sin \tfrac{1}{2}(\phi_L-\phi) \sin^2 \tfrac{1}{2}(\phi_S+\phi)}{\sin \tfrac{1}{2}(\phi_L+\phi)} \left[\frac{\sin \tfrac{1}{2}\left|\phi_K-\phi\right| \sin \tfrac{1}{2}(\phi_E+\phi)}{\sin \tfrac{1}{2}(\phi_K+\phi) \sin \tfrac{1}{2}\left|\phi_E-\phi\right|}\right]^{\beta/\pi} \sin\phi d\phi. \tag{2}$$

Here the constants ϕ_E, ϕ_S, ϕ_L, ϕ_K and c^2 depend on the dimensions of the plate and on α and β. The number c^2 is a proportionality factor determining the scale of the figure. Values of ϕ between 0 and ϕ_L correspond to points on the under side of the plate while values between ϕ_L and π give points on the upper side. The determination of the numbers ϕ_E, ϕ_S, ϕ_L, and ϕ_K is the most difficult part of the whole procedure and will be discussed later. For the present we suppose that they are known. It will be sufficient now to observe that these four constants are angles between 0 and π, whose magnitudes are in the order written, which if used as the upper limit in (2) will yield the positions of the points E, S, L, and K respectively. Also, the angles ϕ_S and ϕ_L are near $\pi/2$ if α and β are small.

If the integration in (2) is performed there is obtained with (1) a complete parametric representation of the speed at the airfoil surface. From Bernoulli's theorem it is then a simple matter to calculate the pressure in terms of the parameter ϕ. If the pressure at infinity is p_∞, the pressure at any point will be

$$p = \frac{\rho}{2}(v_\infty^2 - v^2) + p_\infty \tag{3}$$

where ρ is the density of the fluid.

The total hydrodynamic forces acting on all or part of the airfoil may be found by integration of the pressure. In particular, the total moment of the forces acting on the flap taken with respect to the hinge will be

$$M_H = \int_0^{\phi_E} p(l_F - s) \frac{ds}{d\phi} d\phi - \int_{\phi_K}^{\pi} p(l_F - s) \frac{ds}{d\phi} d\phi, \tag{4}$$

where l_F is the length (chord) of the flap $\overline{ET}$ and moments in the counterclockwise direction are considered positive.

The resultant force on the airfoil is vertical; the drag vanishes and the total lift is (section 9):

$$Y = -\tfrac{1}{2}\pi\rho v_\infty^2 c^2 \left[\sin 2\phi_L - \sin 2\phi_S + \frac{\beta}{\pi}(\sin 2\phi_K - \sin 2\phi_E)\right].$$

In addition to the distributed pressures there is a concentrated force (suction) at the leading edge which acts parallel to the plate LK (see section 10). Its horizontal component must equal the horizontal component of the resultant pressure on the airfoil, since there is no concentrated force at the hinge K. The magnitude of the force at L is

$$F = 4\pi\rho v_\infty^2 c^2 \sin^2 \phi_L \sin^2 \tfrac{1}{2}(\phi_L - \phi_S) \left[\frac{\sin \tfrac{1}{2}(\phi_L + \phi_K) \sin \tfrac{1}{2}(\phi_L - \phi_E)}{\sin \tfrac{1}{2}(\phi_K - \phi_L) \sin \tfrac{1}{2}(\phi_L + \phi_E)}\right]^{\beta/\pi}.$$

The integrations in (2) and (4) must in general be carried out numerically or graphically. The integral in (2) has been carried out in closed form only

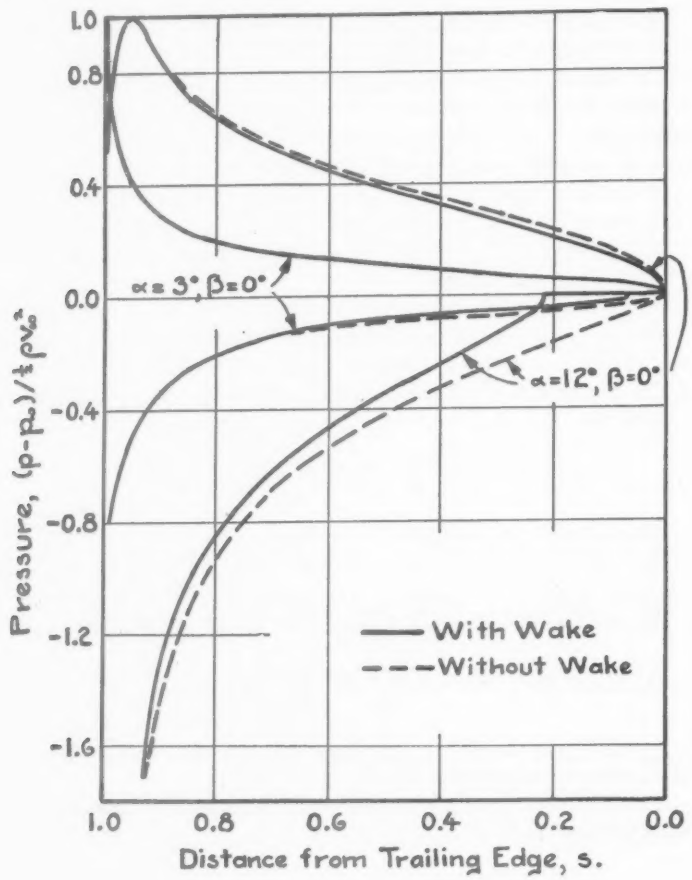

FIG. 2.

for the case of a straight plate, i.e., $\beta=0$, (Schmieden, Reference 4). The problem is much simplified in that case by the fact that of the four constants, ϕ_E and ϕ_K do not occur and $\phi_L=\frac{1}{2}(\pi+\alpha)$, $\phi_S=\frac{1}{2}(\pi-\alpha)$. (See section 4 for the proof).

3. Some numerical results. If the angle of depression of the flap is very small in comparison with the angle of attack, we consider as a first approximation that $\beta=0$. Schmieden (Reference 4) has given a diagram for the pressure

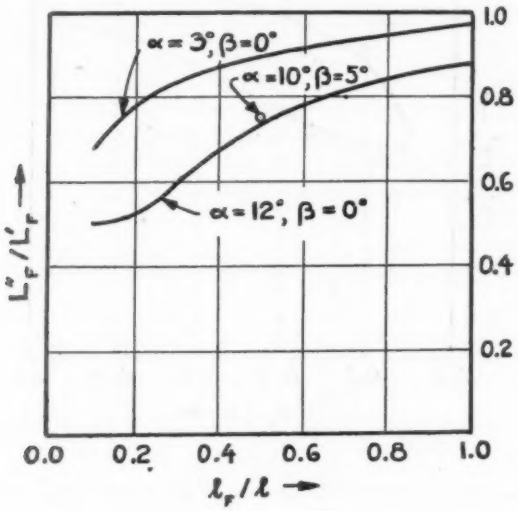

FIG. 3.

L_F' = Total lift on flap without wake, l_F = length of flap = $\overline{KT}$;
L_F'' = Total lift on flap with wake, l = total length of airfoil = $\overline{LK}+\overline{KT}$.

distribution in this case for an angle of attack of 12°, showing a comparison of values obtained both with (Schmieden theory) and without (Kutta-Joukowski theory) a wake. His diagram is reproduced in Fig. 2, together with a similar diagram for an angle of attack of 3°.

Fig. 3 shows the effect of the wake on the total lift L_F of the flap, for flaps of various sizes. Fig. 4 is a similar diagram for the hinge moment M_H of the flap.

Fig. 5 shows the effect of the wake on the pressure distribution for the case $\alpha=10°$, $\beta=5°$ and $\lambda=\overline{KT}/\overline{LK}=1.00$.[5]

[5] For the Kutta-Joukowski solution see: Schmieden, C., *Die Strömung um einen ebenen Tragflügel mit Querruder*, Zeitschrift f. angew. Math. u. Mech. **16**, 194–198 (1936).

See also Ellenberger, G., *Bestimmung der Luftkräfte auf einen ebenen Tragflügel mit Querruder*, Zeitschrift f. angew. Math. u. Mech. **16**, 199–226 (1936).

4. The conditions for the determination of the constants. The Schmieden hypothesis. As ϕ varies from 0 to ϕ_E the corresponding point in Fig. 1 will move along the under side of the flap from T to E; as ϕ goes in succession from ϕ_E through ϕ_S, ϕ_L, ϕ_K to π, the corresponding point will move along

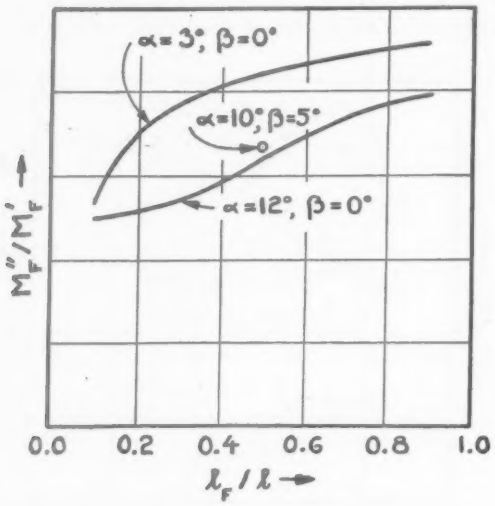

FIG. 4.

M_H' = Hinge moment on flap without wake, l_F = length of flap = $\overline{KT}$,
M_H'' = Hinge moment on flap with wake, l = total length of airfoil = $\overline{LK}+\overline{KT}$.

the under side of the plate through S to L and then back along the upper side through K to D. Now the distance traversed on the under side between E and L must equal the distance on the upper side from L to K. Hence, if the integrand in equation (2) is $f(\phi, \phi_E, \phi_S, \phi_L, \phi_K)$, then

$$\int_E^{\phi_L} f(\phi, \phi_E, \phi_S, \phi_L, \phi_K)d\phi = \int_{\phi_K}^{\phi_L} fd\phi. \tag{5}$$

If $\lambda = \overline{TK}/\overline{KL}$ is the ratio of the length of the flap to the length of the stationary part of the airfoil, we must have

$$\int_0^{\phi_E} fd\phi = \lambda \int_{\phi_E}^{\phi_L} fd\phi. \tag{6}$$

Equations (5) and (6) are the two conditions on the constants imposed by the geometry of the airfoil. A third condition is obtained from the requirement that the velocity must approach the stream velocity as the distance

from the airfoil tends to infinity. The following equation is the mathematical formulation of this condition (for the proof see section 6):

$$\phi_L - \phi_S + \frac{\beta}{\pi}(\phi_K - \phi_E) = \alpha + \beta. \tag{7}$$

For the fourth condition we need the Schmieden hypothesis mentioned

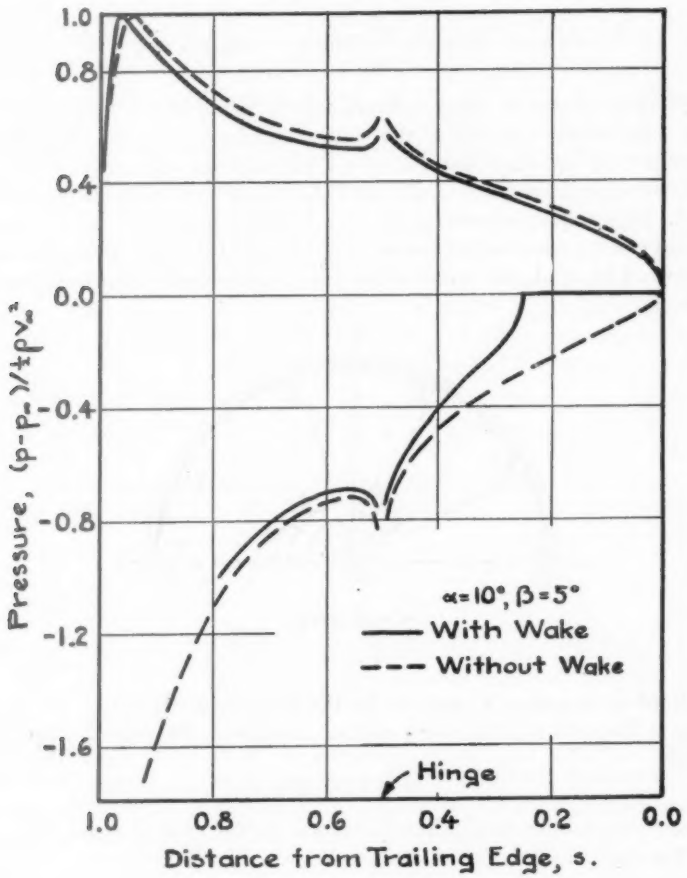

FIG. 5.

in the introduction, or some alternative. The hypothesis which we propose is a natural and direct extension of the physical assumption on the basis of which (7) was obtained. Whatever physical principle causes the velocity to

approach the stream velocity as the distance from the airfoil becomes large might also be expected to cause it to approach the stream velocity *as rapidly as possible.* This is our basic hypothesis; it says that the disturbance at distant points shall be a minimum.

The general mathematical formulation of our hypothesis will be found in section 7, and in section 8 it is shown that for our case of a broken straight line the hypothesis leads to the condition

$$\sin \phi_L - \sin \phi_S + \frac{\beta}{\pi}(\sin \phi_K - \sin \phi_E) = 0. \tag{8}$$

For $\beta = 0$ this reduces to the condition which Schmieden (Reference 4) has used for determining the size of the wake, namely $\phi_L = \pi - \phi_S$. Putting $\beta = 0$ in (7) we get $\phi_L - \phi_S = \alpha$. Hence $\phi_L = \frac{1}{2}(\pi + \alpha)$ and $\phi_S = \frac{1}{2}(\pi - \alpha)$.

The four equations (5), (6), (7), and (8) will determine the four constants involved. These equations can only be solved by successive approximation. Slide rules can be constructed for solving (7) and (8). Equations (5) and (6) can be solved by trial, the integrations being performed either graphically or numerically.

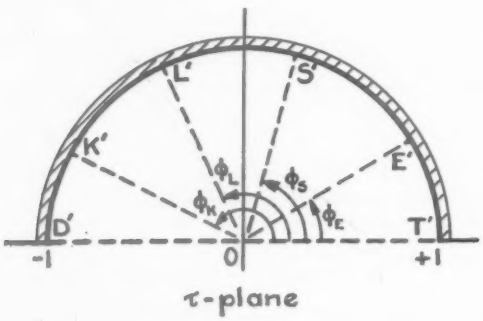

FIG. 6.

5. Proof of formulas (1) and (2) for the velocity distribution. We consider Fig. 1 as a diagram in a complex z-plane, $z = x + iy$. We choose the complex potential

$$w = \psi_1 + i\psi_2$$

in such a way that the streamline $\psi_2 = 0$ will be the critical streamline which follows the surface of the airfoil and wake, with $\psi_1 = 0$ at S.

The w plane minus the positive half of the real axis is mapped onto the interior of the upper half of the unit circle in a τ plane (Fig. 6) by the transformation (for details see, for instance, Reference 4):

$$w = a^2\left(\cos \phi_S - \frac{1}{2}\left(\tau + \frac{1}{\tau}\right)\right)^2. \tag{9}$$

The radii from -1 to 0 and from 0 to 1 in this τ-plane correspond to the free streamlines. The points on the unit circle with arguments ϕ_E, ϕ_S, ϕ_L, and ϕ_K correspond respectively to E, S, L, K. The number a is a constant.

Consider the complex velocity

$$\frac{dw}{dz} = ve^{-i\theta}. \tag{10}$$

Along the real axis in the τ-plane between -1 and $+1$ the speed v must be constant and equal to v_∞, the undisturbed stream velocity. Along the successive segments of the unit circle between the points T', E', S', L', K', and D', the angle θ must have the respective constant values $-(\alpha+\beta)$, $-\alpha$, $\pi-\alpha$, $-\alpha$ and $-(\alpha+\beta)$. It will now be shown that the following expression for the complex velocity will satisfy these conditions:

$$\frac{dw}{dz} = v_\infty e^{i(\alpha+\beta)} \frac{(\tau - e^{-i\phi_L})e^{i\phi_L}(\tau - e^{i\phi_S})}{(\tau - e^{i\phi_L})(\tau - e^{-i\phi_S})e^{i\phi_S}} \cdot \left[\frac{(\tau - e^{-i\phi_K})e^{i\phi_K}(\tau - e^{i\phi_E})}{(\tau - e^{i\phi_K})(\tau - e^{-i\phi_E})e^{i\phi_E}}\right]^{\beta/\pi}. \tag{11}$$

We use that determination of the bracketed quantity to the β/π power whose argument vanishes for $\tau = 1$. The absolute value of this expression is obviously constant and equal to v_∞ if τ is real. For $\tau = e^{i\phi}$ on the unit circle we make use of the identity

$$\begin{aligned} e^{i\phi} - e^{i\phi'} &= e^{\frac{1}{2}i(\phi+\phi')}(e^{\frac{1}{2}i(\phi-\phi')} - e^{-\frac{1}{2}i(\phi-\phi')}) \\ &= 2ie^{\frac{1}{2}i(\phi+\phi')} \sin \tfrac{1}{2}(\phi - \phi') \end{aligned}$$

to rewrite our expression in the form

$$\begin{aligned} \frac{dw}{dz} = v_\infty e^{i(\alpha+\beta)} &\frac{\sin \frac{1}{2}(\phi + \phi_L) \sin \frac{1}{2}(\phi - \phi_S)}{\sin \frac{1}{2}(\phi - \phi_L) \sin \frac{1}{2}(\phi + \phi_S)} \\ &\cdot \left[\frac{\sin \frac{1}{2}(\phi + \phi_K) \sin \frac{1}{2}(\phi - \phi_E)}{\sin \frac{1}{2}(\phi - \phi_K) \sin \frac{1}{2}(\phi + \phi_E)}\right]^{\beta/\pi}, \end{aligned} \tag{12}$$

where the determination is chosen to agree with that in (11). If ϕ is between 0 and ϕ_E each factor involving the sine of a difference is negative, all the negative signs will cancel and the argument of the whole expression will be $\alpha+\beta$; hence $\theta = -(\alpha+\beta)$ as desired. If ϕ is between ϕ_E and ϕ_S, the quantity inside the brackets will be negative and have an argument $-\pi$; hence the entire expression will have an argument α. In this way all the prescribed values of θ may be verified. Since our expression for dw/dz is analytic and not zero inside the upper half of the unit circle in the τ-plane and satisfies the required conditions for v and θ on the boundary of that semicircle, it follows that the flow around our airfoil is thereby correctly determined. If we take absolute values of both sides of (12) we get the required formula (1). Also, from (9), for $\tau = e^{i\phi}$,

$$\frac{dw}{d\tau} = -a^2(\cos\phi_S - \tfrac{1}{2}(\tau + 1/\tau))(1 - 1/\tau^2)$$
$$= -2a^2 i(\cos\phi_S - \cos\phi)e^{-i\phi}\sin\phi$$
$$= 4a^2 i e^{-i\phi}\sin\tfrac{1}{2}(\phi_S + \phi)\sin\tfrac{1}{2}(\phi_S - \phi)\sin\phi. \tag{13}$$

Since $dz = (dz/dw)\,(dw/d\tau)\,d\tau$, we have

$$|\,dz\,| = \frac{1}{v}\left|\frac{dw}{d\tau}\right||\,d\phi\,|. \tag{14}$$

Now ds will have the same sign as $d\phi$ if ϕ is less than ϕ_L and opposite signs if $\phi > \phi_L$. We can therefore get ds from $|dz|$ by removing the absolute value signs around $d\phi$ and around the quantity $\phi_L - \phi$ in the expression (1) for v. Hence (2) follows immediately on substituting (1) and (13) into (14), integrating, and setting $c^2 = a^2/v_\infty$.

6. **Proof of equation (7).** We need only observe that the complex velocity must approach v_∞ as $\tau \to 0$. Putting $\tau = 0$ in (11) we get

$$v_\infty = v_\infty e^{i(\alpha+\beta)}e^{-i\phi_L}e^{i\phi_S}\left[e^{-i\phi_K}e^{i\phi_E}\right]^{\beta/\pi},$$

which reduces at once to (7).

7. **Formulation of the hypothesis of minimum disturbance at distant points.** Consider the function

$$\omega(\tau) = \ln\left(v_\infty\frac{dz}{dw}\right) = \ln\frac{v_\infty}{v} + i\theta. \tag{15}$$

This function is regular within the upper half of the unit circle and continuous on the diameter formed by the segment of the real axis between -1 and $+1$. It is also pure imaginary on this diameter. Hence it can be continued into the lower half of the unit circle by reflection of $i\omega(\tau)$; and $\omega(\tau)$ will then be regular within the entire unit circle.

Expanding $\omega(\tau)$ in a power series we get

$$\frac{dw}{dz} = v_\infty e^{-\omega(\tau)} = v_\infty e^{-\omega(0)-\omega'(0)\tau-\omega''(0)\tau^2/2-\cdots}$$
$$= v_\infty\left\{1 - \omega'(0)\tau + \left[(\omega'(0))^2 - \omega''(0)\right]\frac{\tau^2}{2} - \cdots\right\}. \tag{16}$$

Here we have put $\omega(0) = 0$, which is the condition that the velocity equals v_∞ at infinity.

In all cases discussed in detail by Schmieden and in the present paper, the flow is uniquely determined by the condition

$$\omega'(0) = 0. \tag{17}$$

Hence, there is one and just one flow for which

$$\frac{dw}{dz} - v_\infty \sim \tau^n \quad \text{as} \quad \tau \to 0$$

for $n \geqq 2$; for every other flow $n = 1$. Since $\tau \to 0$ as $z \to \infty$ (for points outside the wake), dw/dz will approach v_∞ at a maximum rate as $z \to \infty$ if the flow satisfies (17). Thus (17) is, for the cases under consideration, the desired mathematical formulation of our hypothesis that the disturbance of the main stream at infinity shall be a minimum.

Equation (17) would be insufficient to determine a unique flow if one of the two points of detachment of the wake had not been determined *a priori*. Thus in the case of an airfoil with sharp trailing edge, one point of detachment is placed at the trailing edge; for the case of a circular cylinder, the two points of detachment are assumed to be symmetrically placed. Situations can be imagined, however, such as an airfoil with blunt trailing edge, where neither point of detachment could be determined in advance. In such a case one can have a still smaller disturbance of the main stream at infinity (i.e. $n \geqq 3$ in the above asymptotic formula) by imposing in addition to (17) the condition

$$\omega''(0) = 0. \tag{17'}$$

For symmetric flows this last condition is automatically satisfied whenever (17) holds.

The theory predicts the following sequence of events as one passes continuously from the case of a circular cylinder to an inclined plate by way of elliptic cylinders with increasing eccentricity but with major axis having a small constant inclination with the direction of the stream: for small eccentricities (and very high Reynolds numbers) both $\omega'(0)$ and $\omega''(0)$ vanish, both the drag and lift are zero (see section 9), and the disturbance of the main stream at distant points is the least possible. As the ellipse becomes flatter and the trailing edge less blunt, the local viscosity and pressure conditions at the trailing edge gain control over the lower point of detachment of the wake, the condition (17′) is lost, and a corresponding lift is developed. The flow in this state corresponds to that in the normal flight of an airfoil. For still flatter ellipses or higher angles of attack or lower Reynolds numbers, the local conditions around the leading edge gain control of the upper point of detachment, the condition (17) is also lost, and there is a positive drag proportional to $\omega'(0)$ and a positive, although much smaller, lift. In this state the airfoil is clearly stalled. Thus the development of a lift and of a drag mark abrupt changes in the state of the flow associated with the loss of the conditions (17′) and (17) respectively; and there is a critical range of Reynolds numbers marking the effect of local conditions which determine the transitions from one type of flow to another.

8. Proof of equation (8). Consider the expression

$$\omega_1(\tau) = \ln(\tau - e^{i\phi_1}) - \ln(\tau - e^{-i\phi_1}),$$

where ϕ_1 is an arbitrary constant angle. Then

$$\omega_1'(\tau) = \frac{1}{\tau - e^{i\phi_1}} - \frac{1}{\tau - e^{-i\phi_1}},$$

$$\omega_1''(\tau) = -\frac{1}{(\tau - e^{i\phi_1})^2} + \frac{1}{(\tau - e^{-i\phi_1})^2},$$

$$\omega_1'(0) = -e^{-i\phi_1} + e^{i\phi_1} = 2i \sin \phi_1,$$

$$\omega_1''(0) = -e^{-2i\phi_1} + e^{2i\phi_1} = 2i \sin 2\phi_1.$$

Now from (11) it follows that $\omega(\tau)$ is the sum of a constant and constant multiples of expressions of the form $\omega_1(\tau)$. Hence

$$\begin{aligned} \omega'(0) &= 2i\left[\sin \phi_L - \sin \phi_S + \frac{\beta}{\pi}(\sin \phi_K - \sin \phi_E)\right], \\ \omega''(0) &= 2i\left[\sin 2\phi_L - \sin 2\phi_S + \frac{\beta}{\pi}(\sin 2\phi_K - \sin 2\phi_E)\right]. \end{aligned} \tag{18}$$

Equation (8) is an immediate consequence of the first of these equations and our hypothesis (17).

9. The total drag and lift. According to the Levi-Civita formula, the total drag on the airfoil is proportional to $\omega'(0)$. From our hypothesis (17) it follows that the drag is zero. In fact, in the case under consideration our hypothesis is equivalent to the condition that the drag vanishes. However, in other situations such as the case of a finite wake discussed by Kolscher (Reference 2), the two conditions differ.

The total lift may be calculated from the Levi-Civita formula

$$Y = \tfrac{1}{4}\pi\rho v_\infty^2 c^2 i\omega''(0), \qquad c^2 = a^2/v_\infty,$$

and (18) to be

$$Y = -\tfrac{1}{2}\pi\rho v_\infty^2 c^2\left[\sin 2\phi_L - \sin 2\phi_S + \frac{\beta}{\pi}(\sin 2\phi_K - \sin 2\phi_E)\right].$$

For $\beta = 0$ this reduces to the Schmieden expression

$$Y = \pi\rho v_\infty^2 c^2 \sin \alpha$$

10. The concentrated forces at L and K. Consider the flow around a corner K with arbitrary angle $\mu\pi$, $0 \leqq \mu \leqq 1$. Draw the small circle C of radius r about K. The finite force at K, if any, will be obtained by applying Euler's momentum theorem to the fluid inside C and taking the limit as r approaches

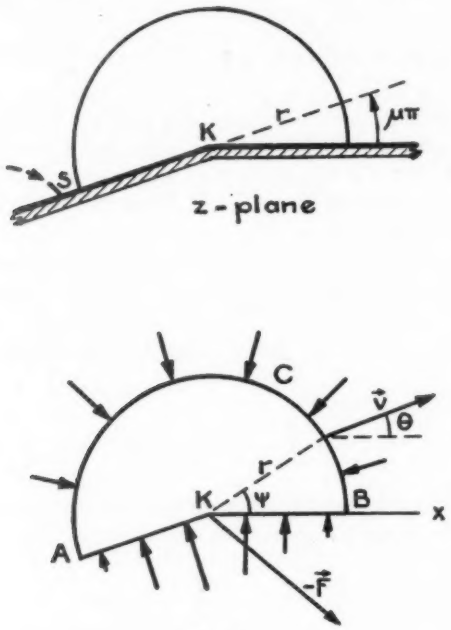

FIG. 7.

zero. In Fig. 7 we show the fluid as a free body, where $-\vec{F}$ is the reaction of the concentrated force $\vec{F}$ on K.

We have

$$\vec{F} - \int_{C+AKB} \vec{n}\, p\, ds = \rho \int_C \vec{n} \cdot \vec{v}\, \vec{v}\, ds,$$

where $\vec{n}$ is the inward unit normal.

It is assumed that the stagnation point S is not at the corner. Hence for small r, setting $\lambda = \mu/(1+\mu)$, $0 \leqq \lambda \leqq \frac{1}{2}$,

$$\frac{dw}{dz} = k/(z - K)^{\lambda} + 0(r^{1-2\lambda}) \tag{19}$$

where k is a constant and $0(r^{1-2\lambda})$ is a quantity such that $0(r^{1-2\lambda})/r^{1-2\lambda}$ is bounded as $r \to 0$. Hence if we put $z - K = re^{i\psi}$,

$$v = |k| r^{-\lambda} + 0(r^{1-2\lambda}), \qquad v^2 = |k|^2 r^{-2\lambda} + 0(r^{1-3\lambda}),$$
$$e^{i\theta} = (|k|/k)e^{i\lambda\psi} + o(r^{1-2\lambda}), \qquad p = -\tfrac{1}{2}|k|^2 \rho r^{-2\lambda} + 0(r^{\sigma}),$$

where $\sigma = 0$ if $\lambda \leqq 1/3$, $= 1-3\lambda$ if $\lambda > 1/3$, and $o(r^{1-2\lambda})$ is a quantity such that $o(r^{1-2\lambda})/r^{1-2\lambda} \to 0$ as $r \to 0$.

Now

$$\int_A^K n\, p\, ds = -\tfrac{1}{2}\vec{n}|k|^2 \rho \int_0^r r^{-2\lambda} dr + \vec{n} \int_0^r 0(r^{\sigma}) dr.$$

Both these integrals tend to zero with r if $\mu < 1$. If $\mu = 1$, this integral and the corresponding integral along KB both diverge, but since, in this case, the pressures along AK and KB are directly opposing, it is permissible to use the principal value of the integral along AKB. With this understanding we may write

$$\int_{AKB} \vec{n}\, p\, ds = -\tfrac{1}{2}\vec{n}_1 |k|^2 \rho \int_0^r (r^{-1} - r^{-1}) dr + \int_0^r \vec{n}\, 0(r^{\sigma}) dr,$$

where $\vec{n}_1$ is the normal to AK. Both these integrals tend to zero. For the arc C,

$$\int_C \vec{n}\, p\, ds = -\tfrac{1}{2}|k|^2 \rho \int_0^{\pi(1+\mu)} r^{-2\lambda+1} e^{i(\psi-\pi)} d\psi + \int_C \vec{n}\, 0(r^{\sigma}) ds. \qquad (20)$$

These integrals also approach zero; the first integral is zero if $\mu = 1$.

The Euler momentum theorem thus reduces to

$$\vec{F} = \lim_{r \to 0} \rho \int_C \vec{n} \cdot \vec{v}\, \vec{v}\, ds.$$

But $\cos(\vec{n}, \vec{v}) = \cos(\psi - \theta - \pi)$, $\vec{v} = ve^{i\theta}$,

$$\rho \int_C \vec{n} \cdot \vec{v}\, \vec{v}\, ds = -\frac{\rho}{2} \int_C v^2 e^{i\psi} ds - \frac{\rho}{2} \int_C v^2 e^{-i\psi} e^{2i\theta} ds.$$

The first of these integrals is similar to (20) and will approach zero for the same reason. For the second integral,

$$-\frac{\rho}{2} \int_C v^2 e^{-i\psi} e^{2i\theta} ds = \frac{-\rho}{2} \int_0^{\pi(1+\mu)} \frac{|k|^2}{r^{2\lambda}} e^{-i\psi} \frac{|k|^2}{k^2} e^{2i\lambda\psi} r d\psi - \frac{\rho}{2} \int_C o(r^{1-4\lambda}) ds.$$

The last integral approaches zero. The first integral also approaches zero if $\mu < 1$. If $\mu = 1$, however, its value is not zero. Evaluating this integral we get finally

$$\vec{F} = -\pi\rho \frac{|k|^4}{k^2} \quad \text{if} \quad \mu = 1,$$
$$= 0 \quad \text{if} \quad \mu \neq 1. \qquad (21)$$

We have thus proved the general result that for a concentrated force to exist at a corner which is not a stagnation point the corner must be a cusp. In particular, there will be no concentrated force at the hinge K. The concentrated force at the leading edge may be found by evaluating k, giving

$$\bar{F} = -4\rho\pi v_\infty^2 c^2 e^{-i\alpha} \sin^2 \phi_L \sin^2 \tfrac{1}{2}(\phi_L - \phi_S)\left[\frac{\sin \frac{1}{2}(\phi_L + \phi_K) \sin \frac{1}{2}(\phi_L - \phi_E)}{\sin \frac{1}{2}(\phi_K - \phi_L) \sin \frac{1}{2}(\phi_L + \phi_E)}\right]^{\beta/\pi},$$

where, as before, $c^2 = a^2/v_\infty$. The direction of the force is parallel to the plate LK and toward the left. In the case $\beta = 0$ this reduces to

$$\bar{F} = -\rho\pi v_\infty^2 c^2 e^{-i\alpha} \sin^2 \alpha,$$

which agrees with the expression given by Schmieden (Reference 4).

11. The width of the wake. It will now be shown that under the minimum hypothesis (17) the width of the wake tends to zero asymptotically as the distance from the airfoil becomes infinite.

We have as in (16)

$$\frac{dz}{dw} = \frac{1}{v_\infty} e^{\omega(\tau)} = \frac{1}{v_\infty}\left[1 + \omega''(0)\,\frac{\tau^2}{2} + \omega'''(0)\,\frac{\tau^3}{6} + \cdots\right].$$

Also

$$\frac{dw}{d\tau} = -a^2\left(\frac{1}{2}\,\frac{1}{\tau^3} - \frac{1}{\tau^2}\cos\phi_S + \cos\phi_S - \frac{1}{2}\,\tau\right).$$

Hence

$$dz = \frac{dz}{dw}\,\frac{dw}{d\tau}\,d\tau = -\frac{a^2}{v_\infty}\left[\frac{1}{2}\,\frac{1}{\tau^3} - \frac{1}{\tau^2}\cos\phi_S + \frac{\omega''(0)}{4}\,\frac{1}{\tau} + 0(1)\right]d\tau.$$

Integrating:

$$z = C_1 + iC_2 - \frac{a^2}{v_\infty}\left[-\frac{1}{4}\,\frac{1}{\tau^2} + \frac{1}{\tau}\cos\phi_S + \frac{\omega''(0)}{4}\ln\tau + 0(\tau)\right],$$

where C_1 and C_2 are real constants. This expression is of the form

$$z = C_1 + iC_2 + \frac{A}{\tau^2} + \frac{B}{\tau} + iD\ln\tau + 0(\tau)$$

where A, B, and D are real constants and A is positive. For τ real and positive this will give the coordinates of a point (x_1, y_1) on the lower boundary of the wake:

$$\begin{aligned} x_1 &= C_1 + \frac{A}{\tau^2} + \frac{B}{\tau} + 0(\tau), \\ y_1 &= C_2 + D\ln\tau + 0(\tau). \end{aligned} \tag{22}$$

If τ is negative we put $\tau = -\tau'$ where τ' is positive and obtain in like manner the coordinates (x_2, y_2) of a point on the upper boundary of the wake:

$$x_2 = C_1 + \frac{A}{\tau'^2} - \frac{B}{\tau'} - \pi D + 0(\tau'),$$
$$y_2 = C_2 + D \ln \tau' + 0(\tau'). \tag{23}$$

Consider the particular values of τ and τ' defined by the equations

$$\frac{1}{\tau} = \frac{-B + \sqrt{B^2 - 4A(C_1 - t)}}{2A},$$
$$\frac{1}{\tau'} = \frac{B + \sqrt{B^2 - 4A(C_1 - \pi D - t)}}{2A},$$

where t is a large real positive parameter. As t becomes infinite both τ and τ' tend to zero through positive values. Substituting these expressions into (22) and (23) we get

$$x_2 - x_1 = 0(\tau') - 0(\tau),$$
$$y_2 - y_1 = D \ln \frac{-B + \sqrt{B^2 - 4A(C_1 - t)}}{B + \sqrt{B^2 - 4A(C_1 - \pi D - t)}} + 0(\tau') - 0(\tau).$$

As $t \to \infty$ both these differences approach zero; hence the width of the wake tends to zero.

12. Details of the calculations. In the numerical calculation of pressure distributions, the evaluation of the constants is the most time consuming. As mentioned previously, it is a simple matter to construct slide rules for Eqs. (7) and (8). It was also found to save time to evaluate the integrals in (5) and (6) graphically. Templates were constructed for the functions $\log \sin \phi$, $\log \sin \phi/2$, and $\beta/\pi \log \sin \phi/2$. By properly adjusting the phase, one can quickly draw from these templates the curves $\log \sin \frac{1}{2}(\phi - \phi_L)$, $\log \sin \frac{1}{2}(\phi + \phi_L)$, $\beta/\pi \log \sin \frac{1}{2}(\phi - \phi_E)$, etc. By addition and subtraction of the ordinates thus obtained one gets the logarithm of the integrand, and hence the value of the integrand itself from a graph of the logarithm function. The integral can then be found either graphically or by means of a planimeter. The pressure distribution shown in Fig. 5 and the associated points in Figs. 3 and 4 were calculated in this way.

If the integrals in (5) and (6) are computed numerically, the infinite discontinuities of the integrand at ϕ_E and ϕ_K are troublesome. There is not actually a discontinuity at ϕ_K, but the integrand goes to zero so abruptly near this point that it is convenient to treat the point as if it were a discontinuity. The usual method of "subtracting off" the discontinuity is unsatis-

factory because the result will be an abrupt zero of the type just mentioned. The following procedure was used as an alternative: after isolating the discontinuity (say at ϕ_1) at the center of a small interval $(\phi_1-\delta, \phi_1+\delta)$, and approximating the sine of an angle by the angle, the integrand reduces to the form $g(\phi)\left|\phi-\phi_1\right|^{\nu}$, where ν is some constant and $g(\phi)$ is continuous (and smooth). A linear transformation will change this integral into one of the form $\int_{-1}^{+1}\bar{g}(\phi')\left|\phi'\right|^{\nu}d\phi'$. One can then approximate $\bar{g}(\phi')$ by a quadratic function of the form $a\phi'^2+b\phi'+c$ by interpolating to the values of $\bar{g}$ at -1, 0, and $+1$; and the resulting integral is easily computed. The final result is a formula for the original integral in terms of the values of g at $\phi_1-\delta$, ϕ_1, and $\phi_1+\delta$ which is strictly analogous to, and in fact is a generalization of, Simpson's rule.

ON A CLASS OF DIFFERENTIAL EQUATIONS IN MECHANICS OF CONTINUA*

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1. Introduction. This paper is concerned with partial differential equations of the form

$$\begin{aligned} u_x &= \tau_1(y)v_y, \\ u_y &= -\tau_2(y)v_x, \end{aligned} \tag{1.1}$$

and with the second order equations

$$\begin{aligned} \frac{u_{xx}}{\tau_1} + \left(\frac{u_y}{\tau_2}\right)_y &= 0, \\ \tau_2 v_{xx} + (\tau_1 v_y)_y &= 0, \end{aligned} \tag{1.2}$$

which are obtained from (1.1) by eliminating either v or u. Here $u=u(x, y)$, $v=v(x, y)$, and subscripts x, y denote partial derivatives. Equations of this form are frequently found in problems of mechanics of continua (cf. the examples in sections 5–7).

The system (1.1) possesses the same structure as the Cauchy-Riemann equations

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned} \tag{1.3}$$

connecting the real and the imaginary parts of an analytic function of $x+iy$. This similarity suggests an integration theory similar in pattern to that of the complex function theory. The fundamentals of such theory are presented in this paper. (A more elaborate mathematical treatment, containing all proofs, will be published elsewhere). The theory will be illustrated by some physical examples. In treating these examples our aim is not to obtain new results in mechanics but rather to present known facts from a simpler and more unified point of view.

In what follows we suppose that the coefficients τ_i $(i=1, 2)$ are *positive analytic* functions of the real variable y. Then the equations (1.2) are of

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elliptic type with analytic coefficients. Therefore u and v will be analytic functions of the real variables x and y.

2. Σ-monogenic functions. If u and v are two conjugate harmonic functions, (i.e. solutions of (1.3)), then $f=u+iv$ is an analytic function of the complex variable $z=x+iy$. From u and v we can obtain two other pairs of conjugate harmonic functions, U, V and u', v', by setting

$$U + iV = \int_{z_0}^{z} f dz, \qquad u' + iv' = f'(z) = \frac{df}{dz},$$

i.e.

$$U = \int_{z_0}^{z} u dx - v dy, \qquad V = \int_{z_0}^{z} v dx + u dy$$

and

$$u' = u_x = v_y, \qquad v' = v_x = -u_y.$$

This procedure can be extended to equations (1.1). Let u and v be a pair of solutions of (1.1). We define the functions

$$U = \int_{z_0}^{z} u dx - \tau_2 v dy, \qquad V = \int_{z_0}^{z} v dx + \frac{u}{\tau_1} dy \tag{2.1}$$

$$u' = u_x = \tau_1 v_y, \qquad v' = v_x = -\frac{u_y}{\tau_2}. \tag{2.2}$$

In (2.1) the integration is extended from an arbitrary fixed point $x_0+iy_0=z_0$ to a variable point $x+iy=z$. By virtue of (1.1) these line integrals do not depend upon the path, but only upon the points z, z_0.

We have

$$U_x = u, \quad U_y = -\tau_2 v, \qquad V_x = v, \quad V_y = \frac{u}{\tau_1},$$

and

$$u'_x = \tau_1 v_{xy}, \quad u'_y = u_{xy}, \qquad v'_x = -\frac{u_{xy}}{\tau_2}, \quad v'_y = v_{xy}.$$

Thus U, V and u, v are solutions of (1.1).

It is convenient to consider

$$f = u + iv$$

as a function of the complex variable $z=x+iy$. If u and v satisfy (1.1) we shall call f a *Σ-monogenic function*, Σ denoting the matrix of the coefficients of (1.1); i.e.,

$$\Sigma = \begin{Vmatrix} 1 & \tau_1(y) \\ 1 & \tau_2(y) \end{Vmatrix}.$$

We have shown that

$$F = U + iV \quad \text{and} \quad f' = u' + iv' \tag{2.3}$$

are also Σ-monogenic functions. We call them the *Σ-integral* and the *Σ-derivative* of f respectively, and write

$$F = \int_{z_0}^{z} f d_\Sigma z, \qquad f' = \frac{d_\Sigma f}{d_\Sigma z}. \tag{2.4}$$

The definition of higher Σ-derivatives is obvious. We write

$$f^{[0]} = f, \quad f^{[1]} = f', \quad f^{[2]} = f'', \quad \text{etc.}$$

A comparison of the formulae (2.1)–(2.4) shows that Σ-integration and Σ-differentiation are inverse processes.

By applying Σ-integration to constants we can obtain an unlimited number of *particular solutions* of (1.1). The function $f=1$ is Σ-monogenic, for $u=1, v=0$ is a solution of (1.1). Therefore

$$Z^{(1)}(z) = \int_0^z 1 d_\Sigma z = x + i \int_0^y \frac{dy}{\tau_1}$$

is Σ-monogenic, i.e., $u=x$, $v=\int_0^y dy/\tau_1$ is a solution of (1.1). Next,

$$Z^{(2)}(z) = 2\int_0^z Z^{(1)} d_\Sigma z = x^2 - 2\int_0^y \tau_2 dy \int \frac{dy}{\tau_1} + 2ix \int_0^y \frac{dy}{\tau_1}$$

is Σ-monogenic and so are

$$Z^{(3)}(z) = 3\int_0^z Z^{(2)} d_\Sigma z, \qquad Z^{(4)}(z) = 4\int_0^z Z^{(3)} d_\Sigma z, \cdots.$$

It is not difficult to find a general formula for $Z^{(n)}(z)$.[1] We set $Z^{(0)}=1$ and define

$$\begin{gathered} Y^{(0)}(y) = 1, \qquad Y^{(1)}(y) = \int_0^y \frac{dy}{\tau_1}, \qquad Y^{(2)}(y) = 2\int_0^y \tau_2 Y^{(1)}(y) dy, \\ Y^{(3)}(y) = 3\int_0^y \frac{1}{\tau_1} Y^{(2)}(y) dy, \cdots. \end{gathered} \tag{2.5}$$

Then

$$Z^{(n)}(z) = \sum_{\nu=0}^{n} \binom{n}{\nu} x^\nu i^{n-\nu} Y^{(n-\nu)}(y). \tag{2.6}$$

[1] Note that the superscript (n) does not denote differentiation. For the gas-dynamical equations written in the form (1.1), where $\tau_1 = 1$, particular solutions corresponding to the formal powers $Z^{(n)}$ have been obtained independently of us by Professor S. Bergman (see footnote 14), and for the imaginary parts by Dr. A. Vaszonyi.

By repeated Σ-integration of the function $f=i$, we obtain another set of Σ-monogenic functions, which we denote by

$$i\cdot Z^{(n)}(z).$$

We have

$$i\cdot Z^{(n)}(z) = i\sum_{\nu=0}^{n}\binom{n}{\nu} x^{\nu} i^{n-\nu} Y^{*(n-\nu)}(y), \tag{2.7}$$

where

$$Y^{*(0)}(y) = 1, \quad Y^{*(1)}(y) = \int_0^y \tau_2 dy, \quad Y^{*(2)} = 2\int_0^y \frac{1}{\tau_1} Y^{*(1)} dy, \cdots. \tag{2.8}$$

Finally, for any complex constant $a=\alpha+i\beta$ we set

$$a\cdot Z^{(n)}(z) = \alpha Z^{(n)}(z) + \beta\{i\cdot Z^{(n)}(z)\}. \tag{2.9}$$

Clearly

$$\frac{d_\Sigma}{d_\Sigma z}\{a\cdot Z^{(n)}\} = n\, a\cdot Z^{(n-1)}. \tag{2.10}$$

From the "formal powers" (2.9) we can construct new particular solutions. Obviously a "formal polynomial" of the nth degree

$$f(z) = a_0 + a_1\cdot Z^{(1)}(z) + \cdots + a_n\cdot Z^{(n)}(z), \qquad a_n \neq 0,$$

is a Σ-monogenic function; i.e., its real and imaginary parts satisfy (1.1). It can be shown that there always exists a formal polynomial of the nth degree which takes prescribed values $A_1, A_2, \cdots, A_{n+1}$ at $n+1$ prescribed points $z_1, \cdots, z_{n+1}$.

A "formal power series"

$$f(z) = \sum_{n=0}^{\infty} a_n\cdot Z^{(n)}(z) \tag{2.11}$$

represents a Σ-monogenic function provided the series converges uniformly and absolutely for sufficiently small values of $|z|$. It can be shown that *any function which is Σ-monogenic around the origin can be represented in the form* (2.11). The coefficients are given by a "Taylor formula":

$$n!a_n = f^{[n]}(0). \tag{2.12}$$

If in defining the formal powers we would extend the integration not from 0 to z but from another fixed point z_0, we would obtain another set of particular solutions, which we denote by

$$a\cdot \underset{z_0}{Z}{}^{(n)}(z).$$

Any Σ-monogenic function defined in the neighborhood of z_0, can be represented in the form

$$f(z) = \sum_{n=0}^{\infty} a_n \cdot Z_{z_0}^{(n)}(z). \tag{2.13}$$

The same method can be applied to the more general equations

$$\begin{aligned} \sigma_1(x)u_x &= \tau_1(y)v_y, \\ \sigma_2(x)u_y &= -\tau_2(y)v_x \end{aligned} \tag{2.14}$$

($\sigma_i > 0, \tau_i > 0, i = 1, 2$). We denote the matrix of the coefficients of (2.14) by Σ,

$$\Sigma = \begin{Vmatrix} \sigma_1 & \tau_1 \\ \sigma_2 & \tau_2 \end{Vmatrix},$$

and call a function $f = u + iv$ Σ-monogenic if u and v satisfy (2.14). The Σ-integral, $F = U + iV$, and the Σ-derivative, $f' = u' + iv'$, of f are defined by

$$U = \int_{z_0}^{z} \sigma_2 u dx - \tau_2 v dy, \qquad V = \int_{z_0}^{z} \frac{v}{\sigma_1} dx + \frac{u}{\tau_1} dy,$$

and

$$u' = \sigma_1 u_x = \tau_1 v_y, \qquad v' = \frac{v_x}{\sigma_2} = -\frac{u_y}{\tau_2}.$$

U, V and u', v' satisfy not (2.14) but the *associated equations*

$$\begin{aligned} \frac{u_x}{\sigma_2(x)} &= \tau_1(y)v_y \\ \frac{u_y}{\sigma_1(x)} &= -\tau_2(y)v_x. \end{aligned} \tag{2.15}$$

Thus F and f' are Σ'-monogenic, where

$$\Sigma' = \begin{Vmatrix} \dfrac{1}{\sigma_2} & \tau_1 \\ \dfrac{1}{\sigma_1} & \tau_2 \end{Vmatrix}.$$

However, $\Sigma'' = \Sigma$, so that the Σ'-integral of F and the Σ'-derivative of f' are Σ-monogenic.

It is clear in which way we must change the definition of the formal powers. We set

$$Z^{(1)} = \int 1 d_{\Sigma} z, \qquad Z^{(2)} = 2\int \tilde{Z}^{(1)} d_{\Sigma} z \quad \text{where} \quad \tilde{Z}^{(1)} = \int 1 d_{\Sigma} z,$$

$$Z^{(3)} = 3\int \tilde{Z}^{(2)} d_{\Sigma} z \quad \text{where} \quad \tilde{Z}^{(2)} = 2\int Z^{(1)} d_{\Sigma} z, \cdots,$$

and similarly for $i \cdot Z^{(n)}$.

Remark. If in (1.1) τ_1 and τ_2 are of ***opposite sign***, the equations (1.2) are of *hyperbolic type*. The definition of Σ-monogenic functions, Σ-integrals and derivatives, formal powers and formal power series remains the same. However, it is *not* true that u and v are necessarily analytic functions of x and y. Neither is it true that *all* Σ-monogenic functions can be represented in the form (2.13).

The integrals defining $Z^{(n)}$ and $i \cdot Z^{(n)}$ must not necessarily converge when τ_1 or τ_2 vanishes at z_0 or at z. If they do converge, they represent Σ-monogenic functions. In this way it is possible to obtain particular solutions of partial differential equations which are of elliptic type in one part of the plane and of hyperbolic in another.

3. Correspondence between Σ-monogenic and analytic functions. Let

$$\varphi(z) = \sum_{n=0}^{\infty} a_n z^n \tag{3.1}$$

be an analytic function of z. We define the Σ-monogenic function

$$f(z) = \sum_{n=0}^{\infty} a_n \cdot Z^{(n)}(z) \tag{3.2}$$

and say that *f corresponds to φ at the origin.*[2]

It can be shown that series (3.2) converges in some neighborhood of the origin. If (3.1) converges everywhere, so does (3.2). If (at the origin) the Σ-monogenic functions f and g correspond to φ and ψ, then $f+g$ corresponds to $\varphi+\psi$, $f^{[n]}$ to $\varphi^{(n)}$, and αf to $\alpha\varphi$, α being a real constant.

The concept of corresponding functions may be of use in discussing physical problems. Assume that some physical phenomenon is described by equations of the form (1.1). Often a simplifying assumption (for instance: the assumption of incompressibility) leads to Cauchy-Riemann equations. Suppose we possess an interesting or typical solution of the simplified problem. It will be given by an analytic function $\zeta = \varphi(z)$. We may expect that the solution of the original more complicated problem given by

$$\zeta = f(z), \ f \text{ corresponding to } \varphi,$$

will be of the same general character.

[2] Thus the formal powers "correspond" to ordinary powers.

We define the Σ-monogenic exponential and trigonometric functions (depending upon a real parameter α)

$$E(\alpha, z), \quad S(\alpha, z), \quad C(\alpha, z), \quad i\cdot E(\alpha, z), \quad i\cdot S(\alpha, z), \quad i\cdot C(\alpha, z) \tag{3.3}$$

as the functions which (at the origin) correspond to the analytic functions

$$e^{\alpha z}, \quad \sin \alpha z, \quad \cos \alpha z, \quad ie^{\alpha z}, \quad i \sin \alpha z, \quad i \cos \alpha z.$$

We have, for instance,

$$E(\alpha, z) = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} Z^{(n)}(z), \qquad i\cdot S(\alpha, z) = \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{2n+1}}{(2n+1)!}\, i\cdot Z^{(2n+1)}(z).$$

The function $C(\alpha, z)+i\cdot S(\alpha, z)$ corresponds to $e^{i\alpha z}$; $E(\alpha, z)+E(-\alpha, z)$, $E(\alpha, z)-E(-\alpha, z)$ correspond to 2 cosh αz and 2 sinh αz.

It is worthwhile to divide the functions (3.3) into their real and imaginary parts. A simple calculation shows that

$$\begin{aligned} E(\alpha, z) &= e^{\alpha x}[c(\alpha, y) + i\, s(\alpha, y)] \\ S(\alpha, z) &= \sin \alpha x\, ch(\alpha, y) + i \cos \alpha x\, sh(\alpha, y) \\ C(\alpha, z) &= \cos \alpha x\, ch(\alpha, y) - i \sin \alpha x\, sh(\alpha, y) \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} s(\alpha, y) &= \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{2n+1}}{(2n+1)!} Y^{(2n+1)}(y), \quad sh(\alpha, y) = \sum_{n=0}^{\infty} \frac{\alpha^{2n+1}}{(2n+1)!} Y^{(2n+1)}(y) \\ c(\alpha, y) &= \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{2n}}{(2n+1)!} Y^{(2n)}(y), \qquad ch(\alpha, y) = \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{(2n)!} Y^{(2n)}(y). \end{aligned} \tag{3.5}$$

Note that the series (3.5) converge for all real values of y and for all complex values of α, so that we may write

$$sh(\alpha, y) = -\, i\, s(i\alpha, y), \qquad ch(\alpha, y) = c(i\alpha, y). \tag{3.6}$$

Similarly we have

$$\begin{aligned} i\cdot E(\alpha, z) &= e^{\alpha x}[-\, s^*(\alpha, y) + ic^*(\alpha, y)] \\ i\cdot S(\alpha, z) &= -\cos \alpha x\, sh^*(\alpha, y) + i \sin \alpha x\, ch^*(\alpha, y) \\ i\cdot C(\alpha, z) &= \sin \alpha x\, sh^*(\alpha, y) + i \cos \alpha x\, ch^*(\alpha, y), \end{aligned} \tag{3.7}$$

where the functions s^*, c^*, sh^* and ch^* are defined in the same way as in (3.5), $Y^{(n)}$ being replaced by $Y^{*(n)}$.

The right hand sides of (3.4) and (3.7) have the form $\Phi_1(x)\Psi_1(y) + i\Phi_2(x)\Psi_2(y)$. Thus we see that the real and the imaginary parts of the Σ-monogenic exponential and trigonometric functions coincide with the functions which we would obtain by solving equations (1.2) by the method of

the *separation of variables*. Therefore the functions (3.3) can be used in solving certain boundary value problems.

The functions $E, \cdots, i \cdot C$ possess many properties analogous to those of ordinary exponential and trigonometric functions. For instance, the Σ-differential equations

$$E^{\backprime} = \alpha E, \qquad S^{\backprime} = \alpha C, \qquad C^{\backprime} = -\alpha S. \tag{3.8}$$

hold.

In the same way we could define correspondence at an arbitrary point z_0 and discuss the functions $\underset{z_0}{E}(\alpha, z), \underset{z_0}{S}(\alpha, z), \cdots$, which at z_0 correspond to the analytic functions $e^{\alpha(z-z_0)}, \sin \alpha(z-z_0), \cdots$.

In some special cases the formal powers and the functions E, S, C, can be expressed in a closed form by means of known functions. Such a case will be discussed in the next section.

4. The case $\tau_1=\tau_2=y^{-p}$. In this section we shall consider the special case

$$\Sigma = \left\| \begin{matrix} 1 & y^{-p} \\ 1 & y^{-p} \end{matrix} \right\|. \tag{4.1}$$

For the sake of simplicity we assume that

$$p \geqq 0,$$

and set

$$p = 2q + 1. \tag{4.2}$$

Along the real axis $\tau_1=\tau_2$ vanishes (except in the trivial case $p=0$). However the integrals defining $Y^{(n)}$ exist, and therefore so do the functions $Z^{(n)}$, E, C, S. The integrals defining $Y^{*(n)}$ converge only if $p<1$. In this case only can we define the functions $i \cdot Z^{(n)}$, $i \cdot E$, $i \cdot C$, and $i \cdot S$.

From (2.5) we obtain by a simple calculation

$$\begin{aligned} Y^{(2\nu+1)}(y) &= \frac{1 \cdot 3 \cdot 5 \cdots (2\nu+1)}{(p+1)(p+3)\cdots(p+2\nu+1)}\, y^{p+2\nu+1}, \\ Y^{(2\nu)}(y) &= \frac{1 \cdot 3 \cdot 5 \cdots (2\nu-1)}{(p+1)(p+3)\cdots(p+2\nu-1)}\, y^{2\nu}, \end{aligned} \tag{4.3}$$

so that, by (4.3) and (2.6),

$$\begin{aligned} Z^{(n)}(z) = n!\Gamma(q+1)\Bigg\{ &\sum_{\nu=0}^{[n/2]} \frac{(-1)^\nu}{\nu!(n-2\nu)!\Gamma(q+\nu+1)}\, x^{n-2\nu}\left(\frac{y}{2}\right)^{2\nu} \\ &+ iy^p \sum_{\nu=0}^{[(n-1)/2]} \frac{(-1)^\nu}{\nu!(n-2\nu-1)!\Gamma(q+\nu+1)}\, x^{n-2\nu-1}\left(\frac{y}{2}\right)^{2\nu+1} \Bigg\}. \end{aligned}$$

Simpler expressions for $Z^{(n)}$ will be found later.

Next we determine the Σ-monogenic exponential and trigonometric functions. For the function s defined by (3.5) we get

$$\begin{aligned} s(\alpha, y) &= \sum_{\nu=0}^{\infty} \frac{(-1)^\nu 1 \cdot 3 \cdots (2\nu+1)}{(2\nu+1)!(p+1)\cdots(p+2\nu+1)} \alpha^{2\nu+1} y^{p+2\nu+1} \\ &= \sum_{\nu=0}^{\infty} \frac{(-1)^\nu \Gamma(q+1)}{2^{\nu}\nu!\Gamma(q+\nu+2)2^{\nu+1}} \alpha^{2\nu+1} y^{p+2\nu+1} \\ &= \tfrac{1}{2}\Gamma(q+1)\alpha y^{p+1} \sum_{\nu=0}^{\infty} \frac{(-1)^\nu \Gamma(q)}{2^{2\nu}\nu!\Gamma(q+\nu+2)} (\alpha y)^{2\nu}. \end{aligned}$$

Recalling the definition of a Bessel function[3] we see that

$$s(\alpha, y) = 2^q \Gamma(q+1)\alpha^{-q} y^{q+1} J_{q+1}(\alpha y). \tag{4.4}$$

Similarly we prove that

$$c(\alpha, y) = 2^q \Gamma(q+1)\alpha^{-q} y^{-q} J_q(\alpha y). \tag{4.5}$$

For $p=0$ (i.e. $q=-\frac{1}{2}$) these formulae transform themselves into the familiar expressions for $I_{1/2}$ and $I_{-1/2}$. For in this case $c(\alpha, y)=\cos \alpha y$, $s(\alpha, y)=\sin \alpha y$.

By (3.4), (3.6), (4.4) and (4.5) we have

$$E(\alpha, z) = 2^q \Gamma(q+1)\alpha^{-q} e^{\alpha x} \{ y^{-q} J_q(\alpha y) + i y^{q+1} J_{q+1}(\alpha y) \}, \tag{4.6}$$

$$S(\alpha, z) = 2^q \Gamma(q+1)(i\alpha)^{-q} \{ y^{-q} \sin \alpha x J_q(i\alpha y) + y^{q+1} \cos \alpha x J_{q+1}(i\alpha y) \}, \tag{4.7}$$

$$C(\alpha, z) = 2^q \Gamma(q+1)(i\alpha)^{-q} \{ y^{-q} \cos \alpha x J_q(i\alpha y) - y^{q+1} \sin \alpha x J_{q+1}(i\alpha y) \}. \tag{4.8}$$

In the case when

$$p > 1$$

the correspondence (at the origin) between analytic and Σ-monogenic functions can be expressed by an integral formula. Since only the $Z^{(n)}$'s are defined we may consider only such analytic functions which are real on the real axis.

For our case the equations (1.2) have the form

$$y\Delta u + p u_y = 0, \tag{4.9}$$

$$y\Delta v - p v_y = 0. \tag{4.10}$$

It is known that the solution of (4.9) which coincides with the analytic function $\varphi(z)$ on the real axis is given by the generalized Laplace integral[4]

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[3] See for instance, E. T. Whittaker and G. N. Watson, *A course of modern analysis*, 3d ed., Cambridge Univ. Press, 1920, p. 359.

[4] See H. Bateman, *Partial differential equations of mathematical physics*, Cambridge University Press, 1932, p. 408.

$$u = \frac{1}{\pi}\int_0^{\pi} \varphi(x + iy\cos\alpha)\sin^{p-1}\alpha\, d\alpha.$$

A simple argument shows that a solution of (4.10), connected with u by (1.1), and vanishing on the real axis is given by

$$v = -iy^p \frac{1}{\pi}\int_0^{\pi} \varphi(x + iy\cos\alpha)\sin^{p-1}\alpha\cos\alpha\, d\alpha.$$

Thus the Σ-monogenic function corresponding (at the origin) to $\varphi(z)$, is given by

$$f(z) = \frac{1}{\pi}\int_0^{\pi} \varphi(x + iy\cos\alpha)(1 + y^p\cos\alpha)\sin^{p-1}\alpha\, d\alpha. \tag{4.11}$$

In particular we have

$$Z^{(n)}(z) = \frac{1}{\pi}\int_0^{\pi} (x + iy\cos\alpha)^n(1 + y^p\cos\alpha)\sin^{p-1}\alpha\, d\alpha. \tag{4.12}$$

The integral (4.11) represents a Σ-monogenic function even when φ possesses a pole at $z=0$. We define

$$Z^{(-n)}(z) = \frac{1}{\pi}\int_0^{\pi} (x + iy\cos\alpha)^{-n}(1 + y^p\cos\alpha)\sin^{p-1}\alpha\, d\alpha. \tag{4.13}$$

It is easily seen that

$$\frac{d_\Sigma}{d_z z} Z^{(-n)}(z) = -nZ^{(-n-1)}(z).$$

We now assume that p is an odd integer. In this case the formal powers can be expressed by Legendre polynomials.

Introducing polar coordinates r, φ,

$$x = r\cos\varphi, \qquad y = r\sin\varphi$$

and recalling that the associated Legendre functions P_n^m admit the representations[5]

$$r^n y^{-m} P_n^m(\cos\varphi) = \frac{1}{\pi}\,\frac{2^m\Gamma(m+1)}{\Gamma(2m+1)}\,\frac{\Gamma(n+m+1)}{\Gamma(n-m+1)}\int_0^{\pi} (x+iy\cos\alpha)^{n-m}\sin^{2m}\alpha\, d\alpha$$

we obtain from (4.12)

$$\operatorname{Re} Z^{(n)}(z) = \frac{\Gamma(p)\Gamma(n+1)}{2^q\Gamma(q+1)\Gamma(p+n)} r^n \sin^{-q}\varphi\, P_{n+q}^q(\cos\varphi). \tag{4.14}$$

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[5] Bateman, l. c., p. 407.

Note that this formula is valid for an arbitrary $p>1$, if P_n^m, where m is not an integer, is defined by Hobson's formula.[6]

If, as we assumed, q is an integer, we have by definition that

$$P_{n+q}^{q}(\cos\varphi) = \sin^{q}\varphi\, P_{n+q}^{(q)}(\cos\varphi)$$

where parenthesis on the superscript indicates differentiation, Then, by (4.13),

$$\operatorname{Re} Z^{(n)}(z) = \frac{(p-1)!n!}{2^q q!(2q+n)!}\, r^n P_{n+q}^{(q)}(\cos\varphi).$$

By (2.2)

$$\operatorname{Im} Z^{(n)} = -\frac{y^p}{n+1}\frac{\partial}{\partial y}\operatorname{Re} Z^{(n+1)},$$

so that by (2.6)

$$\operatorname{Im} Z^{(n)} = \frac{(p-1)!n!}{2^q q!(p+n)!}\, y^p r^{n-1}\{\cos\varphi\, P_{n+q+1}^{(q+1)}(\cos\varphi) - (1+n)P_{n+q+1}^{(q)}(\cos\varphi)\}$$

and

$$Z^{(n)}(z) = \frac{(p-1)!n!}{2^q q!(p+n)!}\, r^n\{(p+n)P_{n+q}^{(q)}(\cos\varphi) + iy^p \sin\varphi[\cos\varphi P_{n+q+1}^{(q+1)}(\cos\varphi) - (n+1)P_{n+q+1}^{(q)}(\cos\varphi)]\}.$$

An analogous formula can be obtained for $Z^{(-n)}$.

If Σ has the form (4.1), particular solutions can be obtained also in another way. Introducing the polar coordinates r, φ as new independent variables, we obtain from (1.1) the system

$$\begin{cases} r^{p+1}u_r = \sin^{-p}\varphi\, v_\varphi \\ r^{p-1}u_\varphi = -\sin^{-p}\varphi\, v_r. \end{cases}$$

Denoting $r+i\varphi$ by ζ and setting

$$\Sigma_* = \begin{Vmatrix} r^{p+1} & \sin^{-p}\varphi \\ r^{p-1} & \sin^{-p}\varphi \end{Vmatrix}, \tag{4.15}$$

we see that a Σ-monogenic function of z is a Σ_*-monogenic function of ζ and vice versa.

Constructing Σ_*-monogenic formal powers, $Z_*^{(n)}(\zeta)$, we obtain particular Σ-monogenic functions of z.

For instance, if $p=0$ (Cauchy-Riemann equations), we have

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[6] Whittaker and Watson, loc. cit., p. 325.

$$\Sigma_* = \begin{Vmatrix} r & 1 \\ \dfrac{1}{r} & 1 \end{Vmatrix}$$

and

$$Z_* = \log z, \qquad i \cdot Z_* = i \log z. \tag{4.16}$$

Of course, this method works not only in the case (4.1) but also when τ_1 and τ_2 are of the form $x^k y^l (x^2+y^2)^m$.

5. Potential flow of an incompressible fluid with rotational symmetry. In this and the following sections we shall briefly consider some mechanical examples leading to equations of the form (1.1).

Perhaps the simplest example is that of a rotationally symmetric potential flow of an incompressible perfect fluid.[7] We introduce cylindrical co-ordinates ρ, θ, z, $\rho = 0$ being the axis of symmetry. Let q_1, q_2, q_3 be the components of the velocity vector $\vec{q}$ in the direction of increasing ρ, θ, z. Then $q_2 = 0$ and $\partial q_1/\partial\theta = 0$, $\partial q_3/\partial\theta = 0$. Using the well known formulae for div $\vec{q}$ and curl $\vec{q}$ we may write the continuity equation in the form

$$\operatorname{div} \vec{q} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho q_1) + \frac{\partial q_3}{\partial z} = 0.$$

The condition of irrotationality takes the form

$$\left| \operatorname{curl} \vec{q} \right| = \left| \frac{\partial q_1}{\partial z} - \frac{\partial q_3}{\partial \rho} \right| = 0.$$

These equations imply the existence of a function u (velocity potential) and of a function v (Stokes' stream function) such that

$$\begin{aligned} q_1 &= u_\rho, & q_3 &= u_z, \\ \rho q_1 &= -v_z, & \rho q_3 &= v_\rho. \end{aligned} \tag{5.1}$$

$u+iv$ may be called the complex potential of the flow. We have, by (5.1)

$$\begin{aligned} u_z &= \frac{1}{\rho} v_\rho \\ u_\rho &= -\frac{1}{\rho} v_z. \end{aligned} \tag{5.2}$$

In order to use our previous notations we shall write x, y instead of z, ρ. Then we see that the complex potential $u+iv$ is a Σ-monogenic function of $x+iy=z$, where

[7] See, for instance, M. M. Munk, *Fluid mechanics*, Part II, in W. F. Durand, *Aerodynamic theory*, Springer, 1934, vol. 1, pp. 260–263.

$$\Sigma = \begin{Vmatrix} 1 & y^{-1} \\ 1 & y^{-1} \end{Vmatrix}.$$

Since Σ is of the form (4.1) we may use the results of section 4. For the formal powers we obtain [cf. (4.2) and (4.14)], when $q=0$, $x=r\cos\varphi$, $y=r\sin\varphi$,

$$Z^{(n)} = \frac{r^n}{n+1}\{(n+1)P_n(\cos\varphi) + iy\sin\varphi[\cos\varphi\, P'_{n+1}(\cos\varphi) - (n+1)P_{n+1}(\cos\varphi)]\}.$$

Using a well known identity this may be written as

$$Z^{(n)} = r^n P_n(\cos\varphi) + \frac{i}{n+1} r^{n-1} y^2 P'_n(\cos\varphi). \tag{5.3}$$

The real and the imaginary parts of the right hand side of (5.3) are the known polynomial solutions of the second order equations for u and v:

$$\begin{aligned} y\Delta u + u_y &= 0, \\ y\Delta v - v_y &= 0. \end{aligned} \tag{5.4}$$

Now we introduce in the x, y-plane polar co-ordinates r, φ (this amounts to introducing spherical co-ordinates in the physical space). Then $u+iv$ will be a Σ_*-monogenic function of $\zeta=r+i\varphi$, where [cf. (4.15)]

$$\Sigma_* = \begin{Vmatrix} r^2 & \sin^{-1}\varphi \\ 1 & \sin^{-1}\varphi \end{Vmatrix}.$$

Forming the first Σ_*-monogenic formal powers we get

$$Z_{1*}^{(1)} = 1 - \frac{1}{r} + i(1-\cos\varphi) = -\left(\frac{1}{r} + i\cos\varphi\right) + \text{const.} \tag{5.5}$$

and

$$\underset{i\pi/2}{i\cdot Z_*^{(1)}} = -\log\tan\tfrac{1}{2}\varphi + ir. \tag{5.6}$$

In the case of a two-dimensional flow (Cauchy-Riemann equations) $Z_*^{(1)}$ was the complex potential of a source and $i\cdot Z_*^{(1)}$ that of a vortex [cf. (4.16)]. Similarly, in our case, (5.5) is recognized as the complex potential of a sink. (5.6) is the potential of a flow for which the streamlines are circles around the origin. The line $y=0$ is a singular line; it is covered by continuously distributed sources and sinks.

Other particular solutions can be obtained by forming higher Σ_*-monogenic powers and by iterated Σ-integration of (5.5) and (5.6).

Σ-differentiating (5.5) we obtain (exactly as in the case of a plane flow) the complex potential of a doublet. In fact this Σ-derivative equals

$$\frac{\cos\varphi}{r^2} - i\,\frac{\sin^2\varphi}{r}.$$

Repeated Σ-differentiation leads to doublets of higher order. It is easily seen that the Σ-monogenic functions obtained in this way coincide with the formal negative power defined in section 4. In our case, $(p=1)$, these powers have the form

$$Z^{(-n)} = r^{-n-1}P_n(\cos\varphi) - iy^2r^{-n-2}P_n'(\cos\varphi).$$

Σ-differentiation of $i\cdot Z_*$ does not lead to new particular solutions, since the Σ-derivative of (5.6) coincides with (5.5) (but for an additive constant).

Finally we note the form of the Σ-monogenic exponential and trigonometric functions. From (4.2) and (4.6)–(4.8) we obtain, setting $p=1$,

$$\begin{aligned} E(\alpha, z) &= e^{\alpha z}\{J_0(\alpha y) + iyJ_1(\alpha y)\} \\ S(\alpha, z) &= \sin\alpha x J_0(i\alpha y) + y\cos\alpha x J_1(i\alpha y) \\ C(\alpha, z) &= \cos\alpha x J_0(i\alpha y) - y\sin\alpha x J_1(i\alpha y). \end{aligned} \tag{5.7}$$

6. Torsion of elastic bodies of revolution.[8] We consider an elastic body of revolution and introduce cylindrical coordinates ρ, θ, z, $\rho=0$ being the axis of symmetry. The physical components of the stress tensor (in the above coordinates) shall be denoted by σ_{ik}, those of the displacement vector by u_i, $i, k=1, 2, 3$. Because of the symmetry these quantities do not depend upon θ. Furthermore, we assume that

$$u_1 = u_3 = 0. \tag{6.1}$$

Then all σ_{ik} vanish except σ_{12} and σ_{23}. The condition of equilibrium takes the form

$$\frac{\partial\sigma_{12}}{\partial\rho} + \frac{\partial\sigma_{23}}{\partial z} + \frac{2\sigma_{12}}{\rho} = 0 \tag{6.2}$$

whereas the stress-strain connection is given by

$$\sigma_{12} = G\,\frac{\partial}{\partial\rho}\left(\frac{u_2}{\rho}\right), \qquad \sigma_{23} = G\,\frac{\partial u_2}{\partial z}. \tag{6.3}$$

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[8] See A. E. H. Love, *A treatise on the mathematical theory of elasticity*, 4th ed., Cambridge University Press, 1934, pp. 330–332, and A. Timpe, *Die Torsion von Umdrehungskörpern*, Math. Annalen **71**, 480–509 (1912).

Substituting from (6.3) into (6.2) we obtain

$$\frac{\partial^2 u_3}{\partial \rho^2} + \frac{\partial^2 u_2}{\partial z^2} + \frac{1}{\rho}\frac{\partial u_2}{\rho} - \frac{u}{\rho^2} = 0. \tag{6.4}$$

If the function ψ is defined by

$$\psi = \rho u,$$

(6.4) may be written as

$$\psi_{\rho\rho} + \psi_{zz} - \frac{1}{\rho}\psi_\rho = 0$$

This equation implies the existence of a function φ connected with ψ by the equations

$$\begin{aligned} \varphi_z &= \frac{1}{\rho}\psi_\rho \\ \varphi_\rho &= -\frac{1}{\rho}\psi_z. \end{aligned} \tag{6.5}$$

φ and ψ can be interpreted as the velocity potential and Stokes' stream function of a rotationally symmetric potential flow [cf. (5.2)]. The particular solutions of (6.5) have been discussed in the preceeding section.

Let our body be a cylinder subjected to a deformation of the kind considered above. If the displacements on the boundary are given, the displacements in the interior can be found by integrating (6.4) under the boundary condition:

$$\psi = \chi(z), \quad \text{for} \quad \rho = \pm \rho_0,$$

ρ_0 being a constant and χ a given function. Plainly, this boundary value problem can be solved by representing χ by a Fourier series (or by a Fourier integral) and by using the particular solutions (5.7). Similarly the solutions (5.7) can be used if the boundary values are prescribed on the lines $z = \text{const.}$ In this case developments in series of Bessel functions should be used. The discussion of convergence will be found in the quoted paper of A. Timpe.

Equation (6.2) can also be written in the form

$$\frac{\partial}{\partial \rho}(\rho^2 \sigma_{12}) + \frac{\partial}{\partial z}(\rho^2 \sigma_{23}) = 0.$$

This equation can be satisfied by introducing a stress function v and setting

$$\rho^2 \sigma_{12} = -\frac{\partial v}{\partial z}, \qquad \rho^2 \sigma_{23} = \frac{\partial v}{\partial \rho} \tag{6.6}$$

On the other hand, writing

$$u = \frac{Gu_2}{\rho},$$

we obtain from (6.3)

$$\sigma_{12} = \rho \frac{\partial u}{\partial \rho}, \qquad \sigma_{23} = \rho \frac{\partial u}{\partial z}. \tag{6.7}$$

Comparing (6.6) and (6.7) we get

$$\begin{aligned} u_z &= \frac{1}{\rho^3} v_\rho \\ u_\rho &= -\frac{1}{\rho^3} v_z. \end{aligned} \tag{6.8}$$

The *torsion problem* consists of integrating equations (6.8) under the boundary condition

$$v = \text{const. on the boundary.}$$

This condition expresses the fact that the surface of revolution is free of stresses.

In order to maintain our previous notations we shall write x, y instead of z, ρ. Then (6.8) shows that $u+iv$ is a Σ-monogenic function of $x+iy=z$, where

$$\Sigma = \left\| \begin{matrix} 1 & y^{-3} \\ 1 & y^{-3} \end{matrix} \right\|.$$

The second order equations for u and v take the form

$$y\Delta u + 3u_y = 0, \tag{6.9}$$

$$y\Delta v - 3v_y = 0. \tag{6.10}$$

Again Σ has the particular form discussed in section 4. From (4.2) and (4.14) we obtain, setting $p=3$,

$$Z^{(n)}(z) = \frac{n!}{(n+3)!} r^n \{(n+3)P'_{n+1}(\cos\varphi) + iy^3 \sin\varphi[\cos\varphi\, P''_{n+2}(\cos\varphi) - (n+1)P'_{n+2}(\cos\varphi)]\}$$

(r, φ are polar coordinates). This is easily transformed into

$$Z^{(n)} = \frac{1}{(n+1)(n+2)} \{r^n P'_{n+1}(\cos\varphi) + ir^{n+3} \sin^4\varphi\, P''_{n+1}(\cos\varphi)\}, \tag{6.11}$$

$$n = 1, 2, \cdots.$$

The imaginary parts of (6.11) are the known polynomial solutions of (6.10).[9]

Next we form the Σ-monogenic exponential and trigonometric functions. From (4.6)–(4.8) we obtain

$$E(\alpha, z) = \frac{2}{\alpha} e^{\alpha x}\left\{\frac{1}{y} J_1(\alpha y) + iy^2 J_2(\alpha y)\right\}$$

$$S(\alpha, z) = \frac{2}{\alpha}\left\{\frac{1}{y} \sin \alpha x J_1(i\alpha y) + y^2 \cos \alpha x J_2(i\alpha y)\right\}$$

$$C(\alpha, z) = \frac{2}{\alpha}\left\{\frac{1}{y} \cos \alpha x J_1(i\alpha y) - y^2 \sin \alpha x J_2(i\alpha y)\right\}.$$

The imaginary part of E again coincides with a well known particular solution of (6.10).[10]

If we introduce the polar coordinates as new independent variables, $u+iv$ becomes a Σ_*-monogenic function of $\zeta = r+i\varphi$, where

$$\Sigma_* = \left\| \begin{matrix} r^4 & \sin^{-3}\varphi \\ r^2 & \sin^{-3}\varphi \end{matrix} \right\|.$$

Forming the first Σ_*-monogenic formal powers we obtain

$$Z_*^{(1)} = 1 - \frac{1}{3r^3} + i\left[\tfrac{2}{3} - \tfrac{1}{3}\cos\varphi\,(\sin^2\varphi + 2)\right] \tag{6.12}$$

$$\underset{i\pi/2}{i \cdot Z_*^{(1)}} = \frac{\cos\varphi}{2\sin^2\varphi} - \tfrac{1}{2}\log\tan\tfrac{1}{2}\varphi + i\frac{r^3}{3}. \tag{6.13}$$

(6.12) describes the torsion of a cone with the vertex at the origin. (6.13) represents the torsion of a sphere around the origin, the torque being transmitted through a singular line along a diameter.

Σ-differentiating (6.12) we get

$$\frac{d_\Sigma}{d_{\Sigma Z}} Z_*^{(1)} = \frac{\cos\varphi}{r^4} - i\frac{\sin^4\varphi}{r}. \tag{6.14}$$

The equations (6.8) can also be interpreted hydrodynamically. $u-iv$ can be understood to be a complex potential [(velocity potential)$+i$(Stokes' stream function)] of a rotationally symmetric potential flow in a five dimensional Euclidean space. (This interpretation is due to Arndt).[11] If we adopt this point of view, $Z_*^{(1)}$ again represents a source, and its Σ-derivative, (6.14),

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[9] Love, l. c., p. 331.

[10] Love, l. c., p. 332.

[11] F. C. Arndt, *Die Torsion von Wellen mit achsensymmetrischen Bohrungen und Hohlräumen*, Thesis, Göttingen, 1916.

a doublet. Combining "flows" of the form (6.12) and (6.14) Arndt solved the torsion problem for a number of bodies with holes.

Repeated Σ-differentiation of (6.14) leads to doublets of higher order. These complex potentials are equivalent to the $Z^{(-n)}$'s defined in section 4.

7. Two dimensional potential gas flow. Let p and ρ represent the pressure and density of a perfect compressible fluid.[12] If heat conduction is neglected the above quantities are connected by the relation

$$p = (p_0/\rho_0^\gamma)\rho^\gamma,$$

where γ is the ratio of the specific heat for constant pressure to the specific heat for constant volume, the subscript zero referring to the fluid at rest. Introducing the local velocity of sound, a, given by

$$a^2 = \frac{dp}{d\rho} = \gamma \frac{p_0}{\rho_0^\gamma} \rho^{\gamma-1},$$

we have by Bernoulli's equation

$$a^2 = a_0^2 - [(\gamma - 1)/2]q^2,$$

where q is the magnitude of the velocity, and

$$\rho = \rho_0 \left[1 - \frac{\gamma - 1}{2} \frac{q^2}{a_0^2}\right]^{1/(\gamma-1)}.$$

The continuity equation for a steady flow has the form

$$\operatorname{div}\left(\frac{\rho}{\rho_0}\vec{q}\right) = 0, \tag{7.1}$$

where $\vec{q}$ is the velocity vector. If the flow is irrotational,

$$\operatorname{curl}\vec{q} = 0. \tag{7.2}$$

For a two-dimensional flow, $\vec{q} = (q\cos\theta, q\sin\theta, 0)$, θ giving the direction of the flow. (7.1) and (7.2) then imply the existence of two functions, $\Phi(x, y)$, $\Psi(x, y)$, such that

$$q\cos\theta = \Phi_x \qquad q\sin\theta = \Phi_y,$$

and

$$q\cos\theta = \frac{\rho_0}{\rho}\Psi_y, \qquad q\sin\theta = -\frac{\rho_0}{\rho}\Psi_x.$$

[12] See, for instance, G. I. Taylor and J. W. MacColl, *The mechanics of compressible fluids*, in W. F. Durand, *Aerodynamic theory*, Springer, 1935, vol. 3, pp. 229–230.

Thus the velocity potential Φ and the stream function Ψ are connected by the equations

$$\Phi_x = \frac{\rho_0}{\rho}\Psi_y$$

$$\Phi_y = -\frac{\rho_0}{\rho}\Psi_x.$$

Since ρ is a non-linear function of the derivatives of the unknown functions Φ and Ψ, the above equations are non-linear. However, linear equations can be obtained by a transformation due to Molenbroek[13] and to Chaplygin;[14] namely, by introducing as new independent variables the quantities θ and q. The equations then become

$$\begin{aligned} \Phi_\theta &= \frac{\rho_0}{\rho} q\Psi_q \\ \Phi_q &= -\frac{\rho_0}{\rho}\left(1 - \frac{q^2}{a^2}\right)\frac{1}{q}\Psi_\theta. \end{aligned} \tag{7.3}$$

To simplify the formulae Chaplygin introduces a new variable

$$t = \frac{2(\gamma - 1)}{\gamma\rho_0^{\gamma-1}} q^2.$$

In what follows we write x, y instead of θ, t. (x, y are *not*, as they were before, the coordinates in the physical plane). Then (7.3) shows that the complex potential $\Phi + i\Psi$ is a Σ-monogenic function of $x+iy$, with

$$\Sigma = \begin{Vmatrix} 1 & \dfrac{2y}{(1-y)^m} \\ 1 & \dfrac{1-(2m+1)y}{2y(1-y)^{1+m}} \end{Vmatrix}, \qquad m = \frac{1}{\gamma - 1}.$$

We see that

$$\tau_1 = 1/\tau_2 = 0 \quad \text{for} \quad y = 0, \qquad \tau_2 = 0 \quad \text{for} \quad y = 1/(2m+1). \tag{7.4}$$

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[13] P. Molenbroek, *Über einige Bewegungen eines Gases mit Annahme eines Geschwindigkeitspotentials*, Arch. d. Mathem. u. Phys. (2). **9**, 157 (1890).

[14] A. Chaplygin, *On gas jets*, Scientific Annals of the Moscow University, Section for Math. and Phys. **21**, 1–121 (1904) (Russian). Cf. Th. von Kármán, *Compressibility effects in aerodynamics*, Journ. Aeronaut. Sc. **8**, 337–356 (1941). (This paper also contains an extensive bibliography.) For a detailed description of the Chaplygin transformation in English see S. Bergman, *The Hodograph Method in the Theory of Compressible Fluids* (suppl. to K. O. Friedrichs, R. v. Mises, *Fluid Dynamics*), mimeographed lecture notes, Brown University, 1942.

The formal powers $a \cdot Z_{z_0}^{(n)}$ cannot be defined for Im $z_0 = 0$. For

$$0 < y < 1/(2m + 1), \tag{7.5}$$

i.e. for *subsonic flow*, we have the elliptic case, $\tau_i > 0$, $i = 1, 2$.

We postpone a systematic discussion of the application of our method to this case and shall show only how an important set of particular solutions (due to Chaplygin) can be represented in terms of our functions.

The second order equation for Ψ takes the form

$$\frac{1 - (2m + 1)y}{2y(1 - y)^{1+m}} \Psi_{xx} + \frac{\partial}{\partial y} \left\{ 2y(1 - y)^{-m} \Psi_y \right\} = 0. \tag{7.6}$$

Chaplygin gives as particular solutions of (7.6)[15]

$$\psi = y^{\alpha} F_{\alpha}(y) \cos \alpha x,$$
$$\chi = y^{\alpha} F_{\alpha}(y) \sin \alpha x,$$

where $\alpha > 0$ and $F_{\alpha}(y)$ is the hypergeometric function

$$F_{\alpha}(y) = F(a_{\alpha}, b_{\alpha}, 2\alpha + 1, y)$$

the constants a_{α}, b_{α} being determined by the conditions

$$a_{\alpha} + b_{\alpha} = 2\alpha - m, \qquad a_{\alpha} b_{\alpha} = -m\alpha(2\alpha + 1).$$

Consider the Σ-monogenic function

$$H = \varphi + i\psi;$$

we have

$$H' = \varphi_x + i\psi_x = 2y(1 - y)^{-m} \psi_y + i\psi_x$$
$$= 2y(1 - y)^{-m} y^{\alpha - 1} [\alpha F_{\alpha}(y) + y F_{\alpha}'(y)] \cos \alpha x - i\alpha y^{\alpha} F_{\alpha}(y) \sin \alpha x.$$

Therefore

$$h = H',$$

satisfies the differential equation

$$\frac{\partial^2 h}{\partial x^2} + \alpha^2 h = 0,$$

i.e. the Σ-differential equation

$$h'' + \alpha^2 h = 0. \tag{7.7}$$

[15] Chaplygin, l. c. Similar solutions have been considered by F. Ringleb, *Exakte Lösungen der Differentialgleichungen einer adiabatischen Gasströmung*, Zeitschr. angew. Math. Mech. **20**, 185–198 (1940). The method by which Chaplygin solved the jet problem and Ringleb computed the complex potentials of a compressible doublet is actually identical with the use of the "correspondence" discussed in section 3.

Let y_0 be a fixed real number, satisfying (7.5). Then

$$h(iy_0) = \alpha A, \qquad h'(iy_0) = i\alpha^2 B \tag{7.8}$$

where

$$\begin{aligned} A &= \frac{1}{\alpha}\, 2y_0(1 - y_0)^m y_0^{\alpha-1}[\alpha F_\alpha(y_0) + y_0 F_\alpha'(y_0)] \\ B &= -\, y_0^{\alpha} F_\alpha(y_0). \end{aligned} \tag{7.9}$$

It is seen easily that the solution of (7.7) with the initial conditions (7.9) is

$$A \underset{iy_0}{C}(\alpha, z) + Bi \cdot \underset{iy_0}{S}(\alpha, z).$$

This function is therefore identical with h. Σ-integrating and suppressing a non-essential constant we obtain

$$H = A \underset{iy_0}{S}(\alpha, z) - Bi \cdot \underset{iy_0}{C}(\alpha, z)$$

so that

$$\psi = \operatorname{Im}\left[A \underset{iy_0}{S}(\alpha, z) - Bi \cdot \underset{iy_0}{C}(\alpha, z)\right].$$

In a similar way we obtain

$$\chi = -\operatorname{Im}\left[A \underset{iy_0}{C}(\alpha, z) + Bi \cdot \underset{iy_0}{S}(\alpha, z)\right].$$

It is interesting to note that if α is a positive integer, $H = \varphi + i\psi$ and $G = \omega + i\chi$ are regular at the origin and that all their Σ-derivatives vanish at this point. This is possible because of (7.4).

We also note that the first formal powers

$$Z^{(1)} \quad \text{and} \quad i \cdot Z^{(1)}$$

represent the complex potentials of a compressible source and vortex respectively.[16]

[16] Cf. Ringleb, l. c.

BOOK REVIEWS

Mathematics of modern engineering. Volume II (Mathematical Engineering). By Ernest G. Keller. (One of a series written in the interest of the Advanced Course in Engineering of the General Electric Company.) John Wiley and Sons, Inc., New York, 1942. xii+309 pp. $4.00.

This book is the second in a series of three, the first of which was written jointly by the present author and R. E. Doherty. Roughly speaking, the present volume treats problems leading to ordinary differential equations while the third volume, apparently, will be devoted to applications of partial differential equations. The material of the book is said to have evolved out of research engineering work and the Advanced Course in Engineering of the General Electric Company. It was prepared for that course and for graduate engineering work in general. It was decidedly not intended either as a textbook in mathematics or in engineering. Its main purpose is to illustrate, by means of many typical examples, the methods of what the author describes as a new phase in engineering and calls Mathematical Engineering.

As in Mathematical Physics, these methods are essentially mathematical in character. Still, mathematical engineering is far from being a part of mathematics. Indeed, solving a practical problem consists of two parts. The formulation of the problem and its reduction to a form in which it lends itself to mathematical treatment constitutes by far the most difficult part. It requires not only knowledge, but also originality and instinct. Once the problem is reduced to a canonical form, its solution may be difficult but can usually be achieved by a well established technique. It should be a comfort to the non-mathematical engineer and to the mathematician alike that, with the numerous engineering examples of the book under review, the second step in each case reduces to routine. In presenting mathematical theories and methods the author has not felt tempted to deviate the least from common usage, thus indirectly paying tribute to the usefulness of "pure" theory.

The great interest, and value, of the book lies in its large collection of examples, taken from the rich experience of the author, and illustrating the application of mathematics to engineering problems. Some of the problems are given as simple exercises (without solutions), others as more or less simple illustrative examples to the mathematical text. The most valuable material is represented by examples where the reader can follow the whole procedure of formulating, and solving, difficult problems of engineering practice. For example, with an intricate problem concerning electric locomotives the factual information is presented in the form in which it once actually confronted the author. The setting up of the problem is done with masterly skill and the final

formulation leads to a surprisingly simple mathematical question. This is an example of mathematical engineering at its best.

The style of the book seems to indicate that the author himself was interested only in presenting this new and interesting material. As a matter of fact, the general theory is treated in a rather unsystematic way. The author carelessly mentions here and there what the needs of the moment require, sometimes altogether forgetting the reader. Apparently, the latter is not even expected to be familiar with vectors and matrices. On the other hand, the short summaries of much more advanced topics cannot seriously be meant as a first introduction. Occasionally the intended elementary character of the book is given up, and once (p. 197) the author almost apologizes for mentioning problems which have already been answered by engineers. One example of the haphazard way in which theory is presented may suffice. Ordinary linear differential equations are first presented as they unfortunately used to be presented in elementary textbooks, without matrices. In connection with G. Kron's theories more than twenty pages are devoted to matrices, and here the said equations are put into matrix form (with the danger of producing the false idea that this is an application of the somewhat mystically formulated first "generalization postulate of G. Kron"). Finally, in the last chapter, for reasons unclear to the reviewer, the same linear equations are treated among non-linear problems, this time in the more efficient matrix form. Now the author also gives good advice on how to compute characteristic values while during the first exposition he suggested impracticable methods. More trivial examples of the same nature appear frequently. Thus, treating the two-dimensional automobile (p. 21), the author says: "Evidently, by Koenig's theorem . . . ." For the theorem itself the reader is referred to p. 44, where he finds only a formula and no theorem. For, only when the author needed the theorem for the second time, did he formulate it in the course of the discussion of a special problem.

The book consists of three parts. Chapter I, "Engineering Dynamics and Mechanical Vibrations" treats applications of classical dynamics. The subject of vibrations seems most congenial to the author. Accordingly, special examples are most numerous in the first chapter; and, on the whole, this chapter seems the best. Chapter II is devoted to an "Introduction to Tensor Analysis of Stationary Neworks and Rotating Electrical Machinery." This is actually an introduction to G. Kron's theories. In the reviewer's opinion this chapter is by far the most difficult one and certainly no introduction in the usual meaning of the word. The exposition is less personal, the examples less numerous and less illustrating. Chapter III is entitled "Non-Linearity in Engineering." It treats briefly of various methods of obtaining solutions, or approximate solutions, of ordinary differential equations. Moreover, elliptic and hyperelliptic functions and integrals are treated and examples of their application to vibration problems are given. There is an extensive bibliography.

but altogether too many of the books mentioned will prove valueless (or confusing) to the average reader.

These deficiencies are not serious enough to endanger the usefulness of the book. They are mentioned in the hope that their removal in a future edition will render the book still more enjoyable and still more useful as a textbook. Even in its present shape the book should convince many a sceptical engineer of the power of mathematical reasoning when applied by a man who combines mathematical knowledge with a good engineering mind.

W. FELLER

Microwave transmission. (International Series in Physics.) By J. C. Slater. McGraw-Hill Book Company, Inc., New York and London, 1942. x+309 pp. $3.50.

This new book in the "International Series in Physics" deals very completely with the theory of transmission of microwaves (electromagnetic waves with wave length of the order of magnitude of 1 cm.). Problems concerning generation and reception of microwaves are omitted except for a few brief remarks, because, as the author points out in the introduction, very little is known from a theoretical point of view about their actual operation.

The treatment of the subject is based largely on electromagnetic theory, although the first chapter is devoted to conventional transmission lines for radio waves. The ideas developed there are kept in mind and used for purposes of comparison in the later chapters. Considerable use is also made of analogies with acoustics and optics. The former are particularly pertinent as the wave length of acoustic waves is of the same order of magnitude as that of microwaves. Much of the terminology introduced is taken by direct analogy from the corresponding acoustic case.

The chapter on the fundamentals of electromagnetic theory is fairly elementary but uses vector notation. Meter, Kilogram, Second (Georgi) units are used throughout. A complete discussion of electrodynamics would of course be out of place in a book of this type. This theory is developed in sufficient detail, however, so that the subsequent chapters should be intelligible to any senior in physics or engineering.

After the preliminaries on ordinary transmission lines and electrodynamics, the author goes on to consider in detail the theory of various types of transmission lines. Wave guides and coaxial cables are thoroughly discussed. Particular emphasis is placed on phenomena which occur when the cross section of a wave guide of cable is varied and which take place at junctions of lines of different characteristic impedance. Problems of matching impedances and the applications are included. The various types of waves that may arise are classified and their propagation characteristics are studied. The theory of radiation from antennas is given in detail, with an entire chapter

devoted to directive devices for antennas. The final chapter deals with the coupling of coaxial lines and wave guides.

The author's treatment of these subjects is extremely good, and his presentation is clear and logical with all extraneous matter eliminated. The book should find considerable use both as a text for the student and as a reference work for the specialist. It represents, moreover, a real contribution, in that its sets forth in concise and unified form the recent advances in microwaves—an extremely important field at the present time. It is an excellent book, to be recommended to all the scientists interested in this field. Those who wish more detailed theoretical calculations should be reminded of Stratton's *Electromagnetic theory* (McGraw Hill, 1941) and E. V. Condon's *Principles of microwave radio* (Reviews of Modern Physics, 14, 341–389 (1942)) where a number of very interesting examples are discussed in full detail in view of practical applications.

L. BRILLOUIN

Introduction to non-linear mechanics. By N. Kryloff and N. Bogoliuboff. A free translation by Solomon Lefschetz of excerpts from two Russian monographs. (Annals of Mathematics Studies, No. 11.) Princeton University Press, Princeton, 1943. iv+105 pp. $1.65.

The major part of this monograph is a very condensed version of a Russian book by the two authors. By making a part of the work of Kryloff and Bogoliuboff accessible to American scientists, the monograph accomplishes an extremely useful purpose and will, no doubt, be welcomed by any one interested in the subject of Non-Linear Mechanics.

The monograph deals mainly with the equation

$$\frac{d^2x}{dt^2} + \omega^2 x = \epsilon f\left(x, \frac{dx}{dt}\right) \tag{1}$$

(for small values of ϵ) and with the problem of finding approximate solutions of (1), free from secular terms and satisfying (1) to within terms of order ϵ^m with prescribed m. Problems studied in the monograph refer, generally speaking, to non-conservative physical systems and differ in this respect from the corresponding problems in the astronomical theory of perturbations.

The mathematical tools are developed mainly from the viewpoint of applications in Physics and Engineering, and mathematical rigor is not the primary consideration. It should also be observed that the qualitative geometrical methods of Poincaré-Bendixon are not discussed in the monograph which deals exclusively with analytical approximation methods.

The first chapter contains various examples in Mechanics and Electrical Circuit Theory which lead to non-linear equations of type (1). The theory of the first approximation is studied in Chapters II and III. Higher approxima-

tions are discussed in Chapter IV. The method of "linearization" described in Chapters V and VI consists in substituting for equation (1) a linear equation which, in the first approximation, has the same solution as (1). The important thing is that this linear equation can usually be set up directly from the physical data of the problem, without reference to the exact equation (1). Chapter VII discusses multiply periodic systems. Periodic disturbances are studied in Chapter VIII. This leads to equations of the type

$$m\frac{d^2x}{dt^2} + kx = \epsilon f\left(t, x, \frac{dx}{dt}\right)$$

where the function f is almost periodic with respect to the variable t. The final chapter contains several excerpts from a monograph by the two authors exhibiting relations between the properties of the exact solutions and those of the approximate solutions.

Throughout the monograph mathematical theories are illustrated by numerous examples of physical systems.

W. HUREWICZ

Heat transmission. By William H. McAdams. (Sponsored by the Committee on Heat Transmission, National Research Council.) Second Edition. McGraw-Hill Book Company, Inc., New York and London, 1942. XIV+459 pp. $4.50.

The knowledge of the processes of the heating and cooling of bodies has gradually accumulated from the dawn of history as a mass of empirical facts. Since the time of Newton who expressed some elementary ideas on the laws of cooling, the thermal aspects of the world have been divided into the equilibrium aspects (thermodynamics) and the non-equilibrium aspects (heat transfer).

All the phenomena of thermodynamics were found to be completely covered by a few sweeping general laws which serve to correlate precisely seemingly unrelated phenomena. No corresponding simple generalizations have been found for the non-equilibrium thermal phenomena. The basic physical laws of conduction and radiation were discovered and applied during the last century and it was clearly understood then that the great majority of heat transfer problems encountered in practice could be adequately treated by these laws only if the attendant convective movements of a fluid were properly accounted for simultaneously. It is the presence of these convective movements which has and still does retard the analytical development of heat transfer knowledge as compared with other aspects of science.

The first edition of *Heat Transmission* by Wm. H. McAdams, Prof. of Chemical Engineering, Mass. Inst. of Tech., appeared in 1933. The emphasis of the book was on the presentation, for ready reference, of all the best empirical data and computational methods required for the solution of prac-

tical heat transfer problems. That the time was ripe for the general collection and presentation of these heat transfer facts is indicated by the fact that the book was sponsored by the Committee on Heat Transfer of the National Research Council. That the book was successful as a text and reference work is attested by the fact that in the 10 years between the first and second editions 9 impressions were needed to satisfy the demand and approximately eight thousand copies were sold.

One quarter of the first edition was devoted to conduction and radiation heat transfer, the physical laws, the physical constants of materials, the empirical and mathematical methods for the solution of problems. The remaining 3/4 of the book presented information needed for the solution of problems involving convection. A chapter on Dimensional Analysis presented the basis of the correlation of empirical data. A chapter on Flow of Fluids presented laminar and turbulent friction and pressure loss data. Succeeding chapters dealt in turn with, the concept of thermal resistance, convection heat transfer by fluids inside pipes, fluids outside pipes, condensing vapors, and boiling liquids. The book closed with some 75 pages of Appendix dealing with thermal properties of materials, Bibliography and Subject Index.

Like most branches of science, the field of heat transfer is one in which developments are being made at an ever more rapid pace. Ten years of work made a considerable revision of the first edition necessary if the book was to maintain its position of excellence in the field of heat transfer literature. During these years, as before the first edition, Prof. McAdams followed carefully the developments as they were made and indeed was himself responsible for some of them.

The new edition has again been sponsored by the Committee on Heat Transfer of the National Research Council.

The book is written in the same spirit as the first edition and follows the same general outline described above. The subject matter of every chapter has however been completely revised. The thoroughness with which the new work in heat transfer has been reported is well indicated by the fact that the former Bibliography of 414 references has been expanded to 789, i.e. almost doubled. These are not useless references either, as each one is followed by a page number of the book where it is mentioned or where its subject matter is pertinent. The new data has been so well assimulated into the old that the book has been increased by only 100 pages.

The opening chapter of the new edition is a four page introduction in which the definitions of conduction, radiation and convection are given together with the basic equation for each. A most important addition is made on page 4 where a table of "Approximate Range of Values of Heat Transfer Coefficients Ordinarily Encountered" is given.

Heat transfer by conduction is covered in 20 pages and as in the earlier

edition presents only the most directly useful facts, in good part through the ideas of a mean area or a thermal resistance. Analytical methods of solution are hardly more than mentioned in passing although some analytical results in the form of resistance formulas are tabulated.

The 17 page chapter on the Heating and Cooling of Solids remains much the same as before. Analytical results, a single sketchy derivation of one case is given, are presented as dimensionless charts, several new ones being included. A graphical method of solving transient heat conduction problems has been added and illustrated by an example.

Radiant Heat Transmission has again been written by H. C. Hottel, a colleague of McAdams on the Chemical Engineering Staff of Mass. Inst. of Tech. The subject matter, 40 pages of it, has been completely revised. The matter of spectral distribution of energy and its effect on heat transfer, hardly more than mentioned in the first edition, is treated in as much detail as the accumulated data on absorptivity and emissivity of bodies will permit. Following the derivation of the general differential equation and a sample integration is a discussion of geometrical and "non-black body" factors. Considerable new results have been added here in the form of what might have been called theorems regarding interrelations between the factors under various circumstances. The radiant heat transfer from gases has likewise been completely rewritten.

The chapter on Dimensional Analysis in the first edition was easily the weakest in the whole book. Considering the fundamental position which this type of analysis holds in the correlation of all convection data, it is important that the fundamental ideas as well as its method of use be made very clear to the reader. The new writing of this chapter is enormously better, in spite of what the reviewer regards as incorrect statements in connection with the "dimensional constants." It is indicated that dimensional constants must be added if more than four dimensions (say M, L, θ, T) are used. Why four? Why not three ($T = ML^2\theta^{-2}$ by kinetic theory)? The answer is simple, kinetic theory as such does not play any part in the heat transfer correlations contemplated in the book. For a similar reason and with the same advantage it is sometimes proper and desirable to use five or six dimensions without adding dimensional constants.

The discussion of Flow of Fluids remains much the same as before except for the addition of some of the dimensionless correlations of velocity distribution in turbulent flow in pipes. The sameness, by chance, even extends to the page numbers for Chapter V; first edition pages 99–134, second edition pages 99–132.

Once the idea of thermal resistance (or conductance) is introduced to permit calculation of series and parallel thermal circuits, the most difficult problem of analysis of heat exchanger performance is the question of "mean

temperature difference." The latest pertinent facts are presented in Chapter VI together with a brief discussion of the important experimental problem of the measurement of surface temperature.

The next four chapters deal with the details of the correlated heat transfer data on fluids inside tubes, outside tubes, condensation and boiling. The revisions and additions are extensive and too numerous to mention in detail. They involve new data used to improve the accuracy of old correlations, new improved methods of correlating old and new data. New theoretical results are mentioned where of practical importance as, for example, in the analogy between heat transfer and friction. Even some new types of heat transfer problems have found an important place, as for example film boiling.

A new last chapter has been added entitled "Applications to Design." In this chapter, structural details, economic considerations and the choice of "optimum" operating conditions are discussed. The criterion of "optimum" is the fixed charges plus operating costs which depend upon the power required to pass the fluids through the heat exchanger. Differentiation gives the conditions for minimum cost which are solved for the optimum conditions sought.

Throughout the book a generous number of illustrative examples are given. As a further aid to the reader, each chapter is concluded with a page of problems on which to test one's real understanding of the subject matter.

The new edition of *Heat Transmission* assures the continuation of the high reputation of the earlier work for another decade or so.

HOWARD W. EMMONS

Empirical equations and nomography. By Dale S. Davis. McGraw-Hill Book Company, Inc., New York and London, 1943. ix+200. $2.50.

The first part of the present book will prove useful to engineers and scientists confronted with the task of finding "a simple mathematical expression fitting some experimental data." The first two chapters are devoted to a discussion of frequently used functions which contain a certain number of constants. Methods of determining the numerical values of these constants are given. The third chapter is concerned with the analysis of experimental data involving two independent variables. It is obvious that this problem cannot be treated comprehensively in the space of some twenty pages allotted to it; the reviewer believes, however, that the very useful graphical method developed by A. Lafay [Génie Civil, 40, 298 (1902)] should have been discussed.

The second part contains an introduction to Nomography. Chapters IV to IX bring the most important types of alignment charts for three and more variables and Chapter X is devoted to special slide rules.

Throughout the book the mathematical argument is of an elementary character; the discussion is illustrated by numerous examples, a large portion of which are concerned with problems of chemical engineering.

W. PRAGER

Electromagnetic Waves. By S. A. Schelkunoff. D. van Nostrand Inc., New York, 1943. xv+530. $7.50.

This new addition to a well-known series has been awaited with much interest by all those acquainted with Dr. Schelkunoff's contributions to propagation theory, and it will be found that their expectations have been entirely fulfilled. This monumental piece of work is equally remarkable for the originality and consistency of its approach as for the wealth of information contained in its five hundred densely packed pages.

The author's systematic use of the harmonic oscillation, with complex variables and coefficients, is in line with the marvelous development which has occurred in the communication field during the last fifty years. Alternating current theory, then acoustics, then vibrational mechanics successively dropped the differential equations which physics offered as a basis and systematically restricted themselves to harmonic oscillations. This has resulted in the replacement of the differential operator by $i\omega$, leading to a tremendous simplification of steady-state analysis, which has been reduced to the calculation of amplitude ratios and phase differences. The genuinely difficult problems have not disappeared for all that but are now relegated to Fourier or Laplace transform theory, and it has become apparent that an enormous field of application can be covered by purely algebraic processes.

Not the least advantage of this method has been the unification brought into the three chapters of technical science mentioned above. Electrical impedances gave the model after which acoustical and mechanical impedances were fashioned; and mixed mutual impedances, thereafter, made it possible to write the equations of electro-mechanical or acoustico-mechanical transducers. There was an exciting era of intense development in this field during the twenties; and it was amusing to hear at that time, and even a good deal later, irate die-hards denouncing "impedances" with bitter irony or viewing with alarm the spread of "analogies."

Dr. Schelkunoff has set about to carry this point of view into Electromagnetic Theory, and it may well be that his will be the honor of having brought into the fold of harmonic oscillation theory the last chapter of Physics which still had to be incorporated. (One might think of Optics, but of course half of the book is really Optics.) Having given, in the first pages of his fourth chapter, a short and quite personal derivation of Maxwell's equations (1–15, p. 69), Dr. Schelkunoff without taking breath adds immediately: "Since we are concerned primarily with fields varying harmonically with time, we replace the instantaneous field intensities and current densities by the corresponding complex variables and write Maxwell's equations as follows:

$$\int E_n ds = -\int\int i\omega\mu H_n dS - \int\int M_n dS,$$
$$\int H_n ds = \int\int (g + i\omega\epsilon) E_n dS + \int\int J_n dS." \qquad (1-16)$$

Thus the sacrosanct Maxwell equations are swept away with movie-like swiftness, and instead we have the steady-state equations of a medium characterized by a distributed series impedance $i\omega\mu$ and a distributed shunt admittance $g+i\omega\epsilon$ (p. 81).

The analogy with a transmission line whose series inductance is μ, shunt conductance g and shunt capacitance ϵ, all taken per unit length, is inescapable (p. 243). In particular the above primary constants simply beg to be transformed into the familiar secondary constants of transmission line theory; here the intrinsic propagation constant σ and the intrinsic impedance η are defined by

$$\sigma = \sqrt{i\omega\mu(g + i\omega\epsilon)}, \qquad \eta = \sqrt{\frac{i\omega\mu}{g + i\omega\epsilon}} \tag{9-1}$$

(p. 81) (σ is in neper/meter, η in ohms; the book is written in MKS—p. 60). For free space we shall have $g=0$, and the following numerical values of the fundamental constants (p. 82):

$$\begin{aligned} &\text{impedance of free space } \eta_0 \approx 120\pi \text{ ohms,} \\ &\text{characteristic velocity } v_0 \approx 3.10^8 \text{ meters/second.} \end{aligned} \tag{9-4}$$

* * * * * * * *

Surprising as it may appear to transmission engineers and sound engineers, who daily handle their respective characteristic impedances Z_0 or ρc, there still are very competent physicists who balk at the idea of free space having a characteristic impedance of about 377 ohms. Yet, in the words of Professor Ronold W. P. King:[1] "The existence of such a characteristic resistance for electromagnetic effects is just as mysterious, but not more so, than the existence of the finite velocity v_0." Dr. Schelkunoff explains very well how this constant could have been overlooked by the builders of the classical theory: "The physicist concentrates his attention on one particular wave: a wave of force or a wave of velocity or a wave of displacement. His original differential equations may be of the first order and may involve both force and velocity; but by tradition he eliminates one of these variables, obtains a second order differential equation in the other and calls it the "wave equation." Thus he loses sight of the interdependence of force and velocity waves . . . " (p. vii). Still, it is surprising to see that one has started with two constants ϵ_0 and μ_0, recognize the fundamental importance of their product, and not enquire about their ratio.

Then, the reader will ask, how can the Theory of Relativity give a leading role to the velocity of light and not mention the impedance of free space. Has Einstein no use for η_0? Well, he has, and he has not. First, an essential point in Special Relativity is the merging of the magnetic and the electric

[1] Mimeographed "Notes on Antennas" for the course of Electronics and Cathode Ray Tubes (Eng. 270), Harvard University.

fields into one skew-symmetrical tensor. When doing this in the MKS system, homogeneity requires the use of the components of E and of $\eta_0 H$; but the factor η_0 is not apparent, for instance, in the equations on p. 44 of "The Meaning of Relativity" (by A. Einstein, Princeton Univ. Press, 1923) which uses a system of units in which $\eta_0 = 1$. Secondly, if we try to connect the universal constant η_0 with other members of this interesting family, we find that η_0 times a (charge)^2 has the dimensions of "action," and more precisely that

$$\eta_0 e^2 = \frac{2h}{137}$$

(e = charge of the electron, h = Planck's constant). We see from this that there is more to η_0 than appears in Special Relativity, the first step in the successive Einsteinian extensions of Maxwell's theory.

* * * * * * * * *

We have dealt at length with this question of the "impedance of free space" because it exemplifies the spirit of the whole work. It occurs in the course of a short but apt presentation of the "Fundamental Electromagnetic Equations" (Chapter IV), immediately applied to harmonic oscillations. The book as a whole is devoted not to Electromagnetism in general but, as specified in the title, to *Electromagnetic Waves*.

Three preliminary chapters introduce the more advanced mathematical tools which will be used, but sparingly, in what follows: such topics as contour integration, Bessel and Legendre functions. Chapter V is a short and original presentation of Network Theory.

The central part of the book begins with Chapter VI, "About Waves in General," a sort of preview of the questions which will be treated in detail later, during which we are introduced to radiation from given currents, propagation along wave guides, and to such general tools as electric and magnetic current sheets, the method of images and conformal representation.

In the following four chapters, we meet the most thorough treatment available of the propagation of waves, guided or bounded, in one, two and three dimensions. It is impossible to do justice here to the richness of the material, which must have cost tremendous labor and which is in great part taken from the author's own publications. We find in Chapter IX, however, classical problems of Fresnel optics, adroitly adapted to contemporary radio needs. Chapter XI is a relatively short treatment of antenna theory, principally of conical antennas, and in the last chapter we return to wave guides and solve various problems involving discontinuities, even to an iris or a transversal wire. This subject is still under development by the author, and the readers of the Quarterly have had the benefit of one of its recent extensions.[2]

[2] See Quarterly of Applied Mathematics, 1, 78 (1943).

The specialist in wave propagation has no need to be told of the value of this book; but the reviewer would like to explain to his fellow non-specialists why it is particularly important that they should not miss it. When the results of much present-day research will suddenly be made available, it will be a hard task to catch up, not only with the new knowledge, but still more with the new modes of attack. The borderland between radio and optics is one of the fields from which great things can confidently be expected. Dr. Schelkunoff's book is a great opportunity for those not at present engaged in research to get familiar with methods which they will want to use tomorrow.

P. LE CORBEILLER

SUGGESTIONS CONCERNING THE PREPARATION OF MANUSCRIPTS FOR THE QUARTERLY OF APPLIED MATHEMATICS

The Editors will appreciate the authors' cooperation in taking note of the following directions for the preparation of manuscripts. These directions have been drawn up with a view toward eliminating unnecessary correspondence, avoiding the return of papers for changes, and reducing the charges made for "author's corrections."

Manuscripts: Papers should be submitted in original typewriting on one side only of white paper sheets and be double or triple spaced with wide margins. The papers submitted should be in final form. Only typographical errors may be corrected on proofs; if authors wish to add material, they may do so at their own expense.

Titles: The title should be brief but express adequately the subject of the paper. The name and initials of the author should be written as he prefers; all titles and degrees or honors will be omitted. The name of the organization with which the author is associated should be given in a separate line to follow his name.

Mathematical work: Only very simple symbols and formulas should be typewritten. All others should be carefully written by hand in ink. Ample space for marking should be allowed above and below all equations. Greek letters used in formulas should be designated by name in the margin. The difference between capital and lower-case letters should be clearly shown; and care should be taken to avoid confusion between zero (0) and the letter *O*, between the numeral one (1) and the letter *l* and the prime ('), between alpha and *a*, kappa and *k*, mu and *u*, nu and *v*, eta and *n*. All subscripts and exponents should be clearly marked, and dots and bars over letters should be avoided as far as possible. Square roots of complicated expressions should be written with the exponent $\frac{1}{2}$ rather than with the sign $\sqrt{}$. Complicated exponents and subscripts should be avoided. Any complicated expression that reoccurs frequently should be represented by a special symbol.

Cuts: Drawings should be made with black India ink on white paper or tracing cloth. It is recommended to submit drawings of at least double the desired size of the cut. The width of the lines of such drawings and the size of the lettering must allow for the necessary reduction. Drawings which are unsuitable for reproduction will be returned to the author for redrawing. Legends accompanying the drawings should be written on a separate sheet.

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